McMaster University

Mathematics 1LS3

Calculus for Life Sciences

All Sections, All Terms
2019/2020

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Solutions to assignments, sample tests and sample exam will be posted on the course web page

IMPORTANT!

The instructor and university reserve the right to modify elements of the course during the term. The university may change the dates and deadlines for any or all courses in extreme circumstances (such as a strike, or a swine flu outbreak). If either type of modification becomes necessary, reasonable notice and communication with the students will be given with explanation and the opportunity to comment on changes. It is the responsibility of the student to check their McMaster email and course websites weekly during the term and to note any changes.

Math 1LS3   2019/20   Course Outline

*** This Course Outline (pages 2-4) is also available on course web page ***

Course web page for Fall 2019 is at ms.mcmaster.ca/lovric/1LS3.html

You are expected to check the web page often, at least after each lecture
(if you think you will not be able to do it, talk to your lecturer as soon as possible)

Course Coordinators and Instructors:

Fall 2019 Coordinator and Instructor: Miroslav Lovric (office: Hamilton Hall 411)
email: lovric@mcmaster.ca
See the course web page for the list of all instructors and their contacts

Lectures AND tutorials are integral parts of the course and you should plan to attend them regularly.

Textbook:

"Calculus for the Life Sciences: Modelling the Dynamics of Life", by F. R. Adler and M. Lovric, Second Canadian Edition, Nelson Education, 2015. ISBN10: 0-17-653078-9. ISBN13: 978-0-17-653078-5

Material covered in the course (selection from the following chapters, not necessarily covered in the order listed):

- Models and Functions; Basic properties of functions, graphs and transformations of graphs (Chapter 1)
- Elementary functions and models (Chapter 2)
- Discrete-time dynamical systems (Chapter 3)
- Limits and basic notions of continuity, as they relate to applications (Chapter 4)
- Describing change: concept of derivative (Chapter 4)
- Rules for calculating derivatives: product, quotient, chain rules, etc. (Chapter 5)
- Working with derivatives and applications of derivatives (Chapters 5 and 6)
- Introduction to differential equations and antiderivatives (Chapter 7)
- Concept of definite integral; Area; Fundamental Theorem of Calculus; Some techniques of integration (Chapter 7)
- Applications of definite integrals (Chapter 7)

Homework assignments: *Although homework assignments are not collected and marked for credit, they are an integral part of the course and you should work on them regularly.*

- Homework questions are included in this booklet
- You are allowed to use any calculator to work on homework questions

- Practice to write well-organized and readable solutions. In particular, you have to justify main steps in your solution: refer to definitions (do not restate them, just identify), rules and known properties. You will learn how to do this by experience (takes time!).
- Suggested completion dates will be announced on the course web page
- Solutions to all assignments will be posted on the course web page

Tests:

- For test dates, consult the course web page
- Details (e.g., material that will be covered, test locations, etc.) will be announced on the course web page about a week before a test
- Tests are written in the evenings; you must bring your student ID to each test
- Standard McMaster calculator Casio fx 991 may be used during tests

Computer labs:

- There will be five computer labs. You do not need to buy anything, software that will be used is free
- All relevant information will be given in class and on the course web page

Final Examination:

- Time/day will be scheduled by the Registrar and announced on the course web page as soon as the information becomes available.
- Details (e.g., material that will be covered, final examination locations, etc.) will be given in class and announced on the course web page
- Standard McMaster calculator, Casio fx 991 MS or MS+ ONLY may be used during final exam

Course Evaluation:

Four tests (all computer labs count as one test) 60 %
Final Exam 40 %

The instructor reserves the right to change the weight of any portion of this marking scheme. For students in good academic standing, other weights might be considered. In either case, the final mark will be computed using this weighting and the new weighting(s). The highest score for a particular student will be her/his final mark. At the end of the course the grades may be adjusted but this can only increase your grade and will be done uniformly. We will use the grade equivalence chart published in the Undergraduate Calendar to convert between percentages and letter grades.

In case of difficulty/problems:

Contact your instructor or your teaching assistant as soon as possible. Failing that, contact the Associate Dean's Office in BSB-129.

Policy regarding missed work:

If you have missed work, it is your responsibility to take action.

If you are absent from the university for a minor medical reason, **lasting no longer than 3 days**, you may report your absence, **once per term, without documentation**, using the McMaster Student Absence Form. Absences for a longer duration or for other reasons must be reported to your Faculty/Program office, with documentation, and relief from term work may not necessarily be granted. When using the MSAF form, report your absence to lovric@mcmaster.ca (in fall term) and to clemene@math.mcmaster.ca (in winter and spring terms). To learn what relief (if any) may be granted for the work you have missed, consult the course web page. Please note that the MSAF **may not be used for term work worth 25% or more**, nor can it be used for the final examination.

Please note! Once a final examination is written, the final grade cannot be adjusted to take into account any special situation.

Academic dishonesty:

You are expected to exhibit honesty and use ethical behaviour in all aspects of the learning process. Academic credentials you earn are rooted in principles of honesty and academic integrity.

Academic dishonesty is to knowingly act or fail to act in a way that results or could result in unearned academic credit or advantage. This behaviour can result in serious consequences, e.g. the grade of zero on an assignment, loss of credit with a notation on the transcript (notation reads: "Grade of F assigned for academic dishonesty"), and/or suspension or expulsion from the university.

It is your responsibility to understand what constitutes academic dishonesty. For information on the various types of academic dishonesty please refer to the Academic Integrity Policy, located at http://www.mcmaster.ca/academicintegrity/
The following illustrates only three forms of academic dishonesty:

1. Plagiarism, e.g. the submission of work that is not one's own or for which other credit has been obtained.
2. Improper collaboration in group work.
3. Copying or using unauthorized aids in tests and examinations.

Your marks:

At the end of the term, all grades in the course will be posted (by student number). It is your responsibility to check for errors before the day of the final exam, and to report any discrepancies to your instructor or to your TA. No error will be corrected unless reported by this time.

Expectations and Objectives

You are expected:

- **to come to classes regularly**, listen and take notes, ask questions, and in general be active [although attendance is not mandatory, you should make an effort and be there; class notes might, or might not be posted on the internet]
- **to study regularly**, work on assignment questions and computer labs and do the work as suggested in lectures and on the course web page
- **to attend and actively participate in tutorials**
- **to do independent work**, such as reading and working through the material in the textbook, or solving exercises, or review high school material that you do not feel comfortable with
- to be responsible about your learning (see *Learning Mathematics* in this booklet and *Learning Resources* in this booklet and on the course web page);
- **as ball-park, you should plan to spend about 2-3 hours studying for every hour of lecture time.**

In this course you will:

- learn about basic concepts of calculus (function, limit, continuity, derivative, integral) and see how they are used in applications in biology, medicine and other life sciences
- learn about mathematical modeling as a tool to introduce mathematical objects and concepts into applications
- learn how to think logically (mathematically) and how to communicate mathematics ideas in writing
- think about mathematics as a discipline (what is a definition? why do we need to prove statements in mathematics? why does mathematics insist on precision and clarity?) and find out how it relates to other areas of human endeavour.

This course will:

- give you a detailed discussion of basic concepts of calculus of functions of one variable
- give you some experience in relating mathematical results obtained using calculus to solutions of problems in other disciplines and to "real-world" problems
- give you experience in constructing and interpreting graphs of functions, so that you will be able to interpret pictorial data obtained from various sources (computers, reference manuals, instruments, various reports, etc.)
- give you experience in reading and writing mathematics, so that you will be able to communicate your mathematical and technical ideas to others and use various reference sources
- give you experience in, and demonstrate benefits of, combining mathematics with computer programming.

Learning Mathematics

Like everything you care for, learning mathematics requires your seriousness, dedication, significant amount of time and hard work.

To learn mathematics means to understand, to memorize and to drill.

To understand something means to be able to correctly and effectively communicate it to somebody else, in writing and orally; to be able to answer questions about it, and to be able to relate it to known mathematics material. Understanding is a result of a mental process. It is not a mere transfer from the one who understands (your instructor) to the one who is supposed to understand (you).

How do you make yourself understand math?

- Ask questions as you study (from your notes, and/or from textbook). Being able to answer questions such as: Why is this true? Why does it work? How does this line follow from the previous line? is a big step towards making yourself understand.
- Discuss what you're studying with your colleagues, teaching assistant or instructor.
- Approach material from various perspectives (draw a graph, make a diagram!).
- Attempt to solve many exercises on your own; as well, study solved problems to see how to write down solutions and answers to questions.
- Make connections with previously taught material and apply what you learnt to new situations.
- It is impossible to understand new mathematics unless one has mastered (to a certain extent) the required background material. Don't skip something just because it is not in your homework assignment.

Keep in mind that building understanding does not happen right away - it takes time, and lots of hard work.

It is necessary to memorize certain mathematics facts, formulas and algorithms. Memorizing is accomplished by exposure: by doing drill exercises, by using formulas and algorithms to solve exercises, by using mathematics facts in solving problems. Remember: the more you understand, the less is left to you to memorize!

The only way to master basic technical and computational skills is to solve a large number of exercises. Drill is essential!

Important Things, in a Nutshell

Come to classes, tutorials and labs regularly.
Be active! Think, ask questions in class, give feedback to your lecturer.

Lecture by itself will not suffice.
You need to spend time on your own doing math: studying, working on assignments, preparing for tests and exams, etc.
Rule of thumb: two to three hours on your own for each hour of lecture.

Plan your study time carefully.
Don't underestimate the amount of time you need to prepare for a test, or to work on an assignment; try not to do everything in the last minute.

Make sure that you are aware of (and use!) learning resources available to you.
Here are some of them:
Lectures, tutorials, and review sessions
Your lecturer's office hours; our teaching assistant's office hours
Math Walk-in Help Centre in Hamilton Hall 104
Student Success Centre

Learn by understanding.
Memorizing *only* will not get you too far. Think, do not just read; highlighting every other sentence in your textbook is not studying!

If you are able to explain something to a colleague and answer their questions about it, then you have learnt it!

Drill is essential for a success (not just in math!).
It's boring, but it works! Solving hundreds of problems will help you gain routine and build confidence you need (together with a few other things) to write good exams.

Calculator …
You are allowed to use any calculator and/or computer software for assignments.
On tests and exams, you will have to use the model that is accepted as a standard at McMaster. The course web page gives detailed information about this.

Homework Assignments

Although homework assignments are not collected and marked for credit, they are an integral part of the course and you should work on them regularly.

For additional practice, if you need it, work on some end of section exercises from your textbook

Practice to write well-organized and readable solutions … in particular, justify main steps in your solution: refer to definitions, theorems, rules and known properties.

There are questions that require the use of a calculator. For homework, any calculator is allowed. On tests and on final exam, you will have to use McMaster standard calculator (this information is posted on the course web page).

Suggested completion dates for assignments will be announced on the course web page. Solutions to all assignments will be posted near the end of the week when the assignments are supposed to be completed (or earlier, if relevant for a test).

Keep in mind that by "studying solutions" (i.e., reading someone else's solutions) you are learning very little (and will forget it soon). You need to work on assignment questions on your own.

Basic Review, Warm-up

1. Compute the following (i.e., express as a single fraction or decimal number), or else say that the expression is not defined.

(a) $(-2)^4$

(b) 2^{-10}

(c) 0^{-3}

(d) $(1/4)^{-3}$

(e) $\sqrt[3]{64}$

(f) $\sqrt[3]{-64}$

(g) $\sqrt{-32}$

(h) $\sqrt{10000}$

(i) $\sqrt{0}$

Continued on next page

2. Factor the following expressions (or else say that it is not possible to do so).

(a) $x^2 - 5$

(b) $4 - a^2$

(c) $x^2 + 1$

(d) $x^3 - 4x$

(e) $x^3 - 1$

(f) $x^3 + 1$

(g) $3x^2 + 4$

Continued on next page

3. Complete the following formulas

(a) $(a+b)^2 =$

(b) $(x-y)^2 =$

(c) $(x+y)^3 =$

(d) $(x-y)^3 =$

(e) $(a-b)(a+b) =$

(f) $(a-b)(a^2+ab+b^2) =$

(g) $(a+b)(a^2-ab+b^2) =$

Continued on next page

4. Draw the graphs of the following functions or equations. Describe the graph in words.

(a) $3x - 2y = 12$

(b) $y = 3$

(c) $x = 7$

(d) $y = x^2$

Continued on next page

(e) $x^2 + y^2 = 4$

(f) $y = x^2 + 1$

(g) $y = x^2 + 2x - 4$

(h) $y = x^3$

Continued on next page

(i) $y = \frac{1}{x}$

(j) $y = -\frac{1}{x}$

(k) $y = \frac{1}{x^2}$

(l) $y = \sqrt{x}$

Continued on next page

5. (a) Solve the equation $3x - 12y = 4$.

(b) Solve the equation $3.2x - 16.2 = -1.3(2 - 3x)$.

(c) Solve the following equation by factoring: $x^2 + 9x + 14 = 0$.

(d) Solve the following equation by factoring: $2x^2 - 5x + 3 = 0$.

Continued on next page

(e) Solve the following equation by factoring: $x^3 - 27x = 0$.

(f) Solve the following equation using the quadratic formula: $x^2 - 6x - 4 = 0$.

(g) Solve the following equation using the quadratic formula: $3x^2 - 4x + 10 = 0$.

(h) Solve the following equation by completing the square: $x^2 - 8x + 11 = 0$.

THE END

Math 1LS3	**Assignment 1**

Sections 1.2, 1.3 (geese) 0.1, 0.2 (elephants): Concepts and Basic Calculations

1. (a) What is a mathematical model? What does it consist of?

(b) What is a discrete-time dynamical system? What is a continuous-time dynamical system?

(c) Identify each interval as open or closed: $(1, 4)$, $(-\infty, 7]$, $[1, 4]$, $(-\infty, 7)$, $(0, \infty)$.

(d) What is an order of magnitude? By how many orders of magnitude is 450,000 larger than 37?

(e) What does the symbol $\propto$ in $M \propto 1/K$ mean?

Continued on next page

2. (a) How many seconds are there in a year?

(b) Assuming that a human heart beats 72 times per minute, calculate how many times it is going to beat in 50 years.

(c) Find the mass in kilograms of a spherical object whose diameter is 15 inches and density is 3.4 g/cm^3. [Hint: density=mass/volume]

(d) A polar bear can weigh as much as 650 kg. Express its weight in grams and in pounds (lb).

(e) The fastest land animal (cheetah) can reach the speed of 95 km/h. Convert its speed into miles per hour (mph).

Continued on next page

3. (a) What is a parameter? In Exercise 1 on page 22 (geese) 19 (elephants), identify independent variable, dependent variable and a parameter.

(b) What is a relation? What is a function? Given that $g(m) = 32m^2d + 16\sqrt{m}$, which symbol represents the independent variable? Dependent variable? Parameter?

(c) What is the purpose of the vertical line test?

(d) What is the natural domain of a function? What is the given domain of a function?

(e) What is the graph of a function?

Continued on next page

4. Consider the following functions: $f_1(x) = 3x - 5$, $f_2(x) = -6x$, $f_3(x) = 11$, $f_4(x) = x^2$, $f_5(x) = x^3$, $f_6(x) = 1/x$, $f_7(x) = 1/x^2$, $f_8(x) = \sqrt{x}$, $f_9(x) = \sqrt[3]{x}$, $f_{10}(x) = |x|$.

From this list identify all functions:
(a) whose domain is the set of all real numbers.

(b) which are not defined at $x = 0$.

(c) whose graphs are hyperbolas.

(d) which are increasing for all real numbers.

(e) which are decreasing on the interval $(-\infty, 0)$ only.

(f) which are linear.

(g) which are positive (i.e., whose range consists of positive numbers).

(h) which are symmetric with respect to the origin.

(i) which are symmetric with respect to the y-axis.

(j) which are constant.

Continued on next page

5. (a) Sketch the graph of the function

$$h(x) = \begin{cases} x^3 - 1 & \text{if} \quad x < 1 \\ 0 & \text{if} \quad 1 \leq x \leq 2 \\ 1/x & \text{if} \quad x > 2 \end{cases}$$

and identify its domain and range.

6. Find the domain of the function $f(x) = \sqrt{x^2 - 1}$.

Continued on next page

7. Find the domain of each function.

(a) $f_1(x) = \dfrac{x+1}{x-1}$

(b) $f_2(x) = \dfrac{x+1}{x^2-1}$

(c) $f_3(x) = \dfrac{x+1}{x^2+1}$

(d) Sketch the graph of the function $f_2(x)$ from (b). Looking at the graph, determine the range of $f_2(x)$.

Continued on next page

8. Draw graphs based on the following descriptions.

(a) A population of fish begins at a small value, quadruples, remains stable (i.e., constant) for some time, and then decreases to half of the initial population.

(b) Sea temperature fluctuates between high values during the day and low values at night, the difference between high and low temperatures being about 10 degrees C.

(c) A population of birds starts at a large value, decreases to a very small value, and then increases to an intermediate value.

9. Question 40 on page 24 (geese) 21 (elephants) in the textbook.

Continued on next page

10. Let $f(x) = 3x^2 - x + 2$. Find (and simplify as much as possible)
(a) $f(a+1)$

(b) $f(a) + f(1)$

(c) $f(1+h) - f(1)$

11. Let $f(x) = \dfrac{2}{x}$. Find (and simplify as much as possible)
(a) $f(a/4)$

(b) $f(22/a)$

(c) $\dfrac{f(3+h) - f(3)}{h}$

Section 1.4 (geese) 0.3 (elephants): Working with functions

1. (a) Define the composition $m \circ p$ of functions m and p.

(b) Look at Example 1.4.2 (geese) 0.3.2 (elephants). What can you say about the product of an increasing function and a decreasing function?

(c) What is the purpose of the horizontal line test? State the test.

(d) Define the term: inverse function.

(e) Write down the cancellation formulas for a function and its inverse.

Continued on next page

2. What is an algebraic function? Transcendental function?

3. Given is the graph of $f(x)$. Explain (in words) how to obtain the graph of

(a) $f(x - 13)$

(b) $2f(x) - 13$

(c) $2f(x - 13)$

(d) $-4f(x + 13)$

(e) $f(3x) + 1$

(f) $-f(x/10)$

Continued on next page

(g) $f(-x)/4$

(h) $-4f(-x)$

4. Read Example 1.4.17 (geese) 0.3.17 (elephants) and use it to sketch the graph of the function $f(x) = |x^2 - 1|$.

5. Exercise 48 on page 45 (geese) 40 (elephants).

Continued on next page

6. (a) Sketch the graph of the function $y = 1/x$.

(b) Starting with the graph in (a), sketch the graph of the function $y = -\frac{1}{x}$.

(c) Starting with the graph in (a), sketch the graph of the function $y = \frac{2}{x}$.

(d) Starting with the graph in (a), sketch the graph of the function $y = \frac{1}{x-4} + 7$.

Continued on next page

7. Given that $f(x) = 2x + 3$, $g(x) = x^2 - 1$, and $h(x) = \frac{1}{x-2}$. Compute the following compositions

(a) $(f \circ g)(x)$ and $(g \circ f)(x)$. Is $(f \circ g)(x) = (g \circ f)(x)$?

(b) $(g \circ h)(x)$ and $(h \circ g)(x)$. Is $(g \circ h)(x) = (h \circ g)(x)$?

Note: above two examples illustrate that the composition of two functions is not commutative; i.e., in general, for two functions f and g, $f \circ g \neq g \circ f$.

(c) Evaluate $(f \circ h)(3)$.

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8. (a) Let $f(x) = x^{-1}$ and $g(x) = (x^2 - 1)^{-3}$. Express the compositions $(f \circ g)(x)$ and $(g \circ f)(x)$ as fractions.

(b) Let $f(x) = \sqrt{2 - x}$ and $g(x) = \sqrt{x - 2}$. Find the composition $(g \circ f)(x)$.

(c) Let $f(x) = 12 - x^2$ and $g(x) = 4$. Find the compositions $(f \circ g)(x)$ and $(g \circ f)(x)$.

Continued on next page

9. (a) Exercise 4, page 44 (geese) 39 (elephants) in the textbook.

(b) Exercise 6, 44 (geese) 39 (elephants) in the textbook.

10. Which of the functions $f_1(x) = 3x - 5$, $f_2(x) = -6x$, $f_3(x) = 11$, $f_4(x) = x^2$, $f_5(x) = x^3$, $f_6(x) = 1/x$, $f_7(x) = 1/x^2$, $f_8(x) = \sqrt{x}$, $f_9(x) = \sqrt[3]{x}$, $f_{10}(x) = |x|$ pass the horizontal line test?

Continued on next page

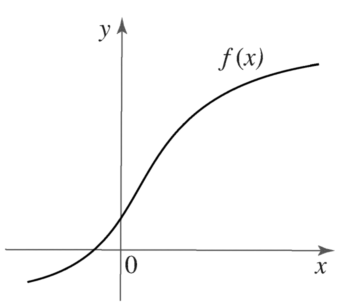
11. Find $f^{-1}(x)$.

(a) $f(x) = \sqrt{x} + 4$.

(b) $f(x) = \dfrac{1-x}{2+x}$.

(c) $f(x) = (x-4)^7$.

(d)



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12. The function $m(t)$ gives the number of monkeys on a island, where t is measured in years. The value $t = 0$ represents the year 2000 (thus $t = 1$ is the year 2001, etc.). The values of f are: $m(0) = 15$, $m(1) = 23$, $m(2) = 33$, $m(3) = 37$, $m(4) = 41$, $m(5) = 47$, $m(6) = 65$, $m(7) = 68$, $m(8) = 84$, $m(9) = 92$, $m(10) = 97$.

(a) Explain why m has an inverse function. Explain in words what the function m^{-1} represents. [Hint: read Example 1.4.6 (geese) 0.3.6 (elephants)]

(b) Find $m^{-1}(65)$ and explain what the answer means.

(c) The function $c(m)$ gives the number of coconuts eaten by m monkeys. What does the composition $(c \circ m)(t)$ represent?

THE END

Math 1LS3	Assignment 3

Section 2.2 (geese) 1.2 (elephants): Exponential and logarithm functions

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1. Use laws of exponents to simplify the following expressions (as much as possible without a calculator).

(a) $\dfrac{1.6^2}{1.6^{-3}}$

(b) $\dfrac{(21.6^2)^4}{21.6^6}$

(c) $\dfrac{(1.6^2)^4}{1.6^8}$

(d) $3.3^{0.44} \cdot 3.4^{-0.46} \cdot 3.3^{0.44} \cdot (3.4^{0.23})^2$

(e) $\dfrac{(3.79^{1/2})^7}{\sqrt{3.79^4}}$

(f) $3^3 \cdot 9^2 \cdot 27$

Continued on next page

2. (a) Sketch the graphs of $y = 2^x$, $y = e^x$ and $y = 3^x$ in the same coordinate system.

(b) Sketch the graphs of $y = e^x$, $y = e^{2x}$ and $y = e^{-x}$ in the same coordinate system.

(c) Sketch the graphs of $y = e^{0.3x}$ and $y = -2e^{0.3x}$ in the same coordinate system.

Continued on next page

3. Use laws of logs to simplify (if possible) the following expressions (as much as possible without a calculator).

(a) $\log_{10} 10000$

(b) $\log_{10} 10^{-0.33}$

(c) $\log_{10} 10$

(d) $\log_{10}(22 + 78)$

(e) $\log_{10} 50 + \log_{10} 2000$

(f) $\log_{10} 0.001 + \log_{10} 0.1$

(g) $\log_{10} 0.001 - \log_{10} 0.1$

(h) $\ln e^{0.09}$

(i) $\ln(1/e)$

(j) $\ln(e + 11.4)$

(k) $\ln 11.4e$

Continued on next page

4. (a) Sketch the graphs of $y = \log_{10} x$ and $y = \ln x$ in the same coordinate system.

(b) Sketch the graphs of $y = \log_{10}(x + 2)$, $y = \log_{10} x + 2$ and $y = 3 \log_{10} x$.

(c) Sketch the graphs of $y = 2 \ln x$, $y = -2 \ln x$ and $y = 1 - 2 \ln x$.

Continued on next page

5. Solve the following equations.

(a) $4e^x = 7$

(b) $4e^{5x-1} = 64$

(c) $0.5^{x^2} = 0.125$

(d) $\ln(2 - 3x) = 3.5$

(e) $\ln(x + 6) = 2 \ln x$

(f) $\ln(x - 4) + \ln 3.5 = \ln 7$

Continued on next page

6. A population of fish is given by $P(t) = 3.2 \cdot 10^4 \cdot 1.32^t$. When will the population reach 1 million? 1 billion?

7. Find the doubling time of the population that behaves according to the equation $P(t) = 1200e^{1.32t}$.

Continued on next page

8. A radioactive substance lost 2/3 of its mass in 1 year. Find its half-life (in years).

9. A bacterial culture starts with 100 bacteria, and after two hours the population is 250 bacteria. Assuming that the rate of growth of the population is proportional to its size, find the number of bacteria after 4 hours.

Continued on next page

10. Suppose that a radioactive substance decays according to $M(t) = 24.5e^{-0.0032t}$, where t is time in years.

(a) How long will it take for the substance to reach 45% of its original amount? 44% of its original amount? What is the time difference?

(b) How long will it take for the substance to reach 10% of its original amount? 9% of its original amount? What is the time difference?

(c) In both (a) and (b) the substance loses 1% of its amount. Why aren't the answers in (a) and (b) the same?

Section 2.3 (geese) 1.3 (elephants): Trig and inverse trig basics

1. (a) Convert 210 degrees into radians.

(b) Convert $-9\pi/2$ radians into degrees.

(c) Draw the angles whose measure is $4\pi/3$, $5\pi/3$, and $7\pi/3$.

(d) Given that $\cos\theta = 2/3$, $0 < \theta < \pi/2$. Find the remaining trigonometric ratios.

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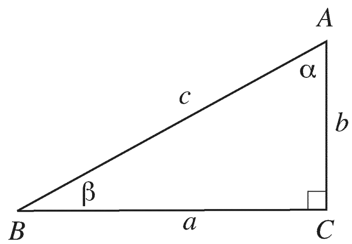
2. Let ABC be a right triangle, where $\angle C = 90^0$; see the figure below.



(a) Given that $a = 7$ and $b = 4$, find $\sin\alpha$, $\cos\alpha$, $\sin\beta$, and $\cos\beta$.

(b) Given that $\cos\beta = 12/13$ and $c = 13$, find a, b, $\sin\beta$, and $\tan\beta$.

(c) Given that $c = 1$ and $a = 0.6$, find all six trigonometric ratios for angle β.

Continued on next page

3. (a) Find the values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for $\theta = \pi/4$.

(b) Find the values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for $\theta = 3\pi/4$.

(c) Find the values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for $\theta = \pi/3$.

(d) Find the values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for $\theta = 2\pi/3$.

Continued on next page

4. (a) Sketch the graphs of $y = \cos x$ and $y = \cos 2x$ in the same coordinate system.

(b) Sketch the graphs of $y = \sin x$ and $y = \sin(x/3)$ in the same coordinate system.

(c) Starting with the graph of $y = \tan x$, sketch the graph of $y = \tan(x + \pi/2)$.

Continued on next page

5. Graph the following functions, identify and mark on your graph the following quantities: maximum, minimum, average, amplitude, period and phase shift.

(a) $f(x) = 3.2 + 0.8\sin(x - \pi)$

(b) $f(x) = 3.2 + 0.8\sin(2x - \pi)$

Continued on next page

6. (a) Find all solutions of the equation $\sin x = 1$ such that $0 \leq x \leq 2\pi$.

(b) Find all solutions of the equation $\sin x = 1$.

(c) Find all solutions of the equation $\cos x = 0$ such that $0 \leq x \leq 2\pi$.

(d) Find all solutions of the equation $\cos x = 0$.

Continued on next page

7. (a) Find all solutions of the equation $\sin x = 1/2$ such that $0 \le x \le 2\pi$.

(b) Find all solutions of the equation $\sin x = 1/2$.

(c) Find all solutions of the equation $\cos x = -1$.

(d) Find all solutions of the equation $\tan x = 1$.

Continued on next page

8. (a) Rewrite $\sin(\pi/2) = 1$ using arcsin function.

(b) Compute $\arcsin(\sin 0)$ and $\arcsin(\sin \pi)$. Is it true that $\arcsin(\sin x) = x$ for all x?

(c) Find $\arccos 1$ and $\arctan 1$.

(d) Find $\arcsin(-1)$ and $\arctan(-1)$.

(e) Explain why $\arcsin 2$ is not defined.

THE END

Section 1.2 (geese) 0.1 (elephants) Conversion between units

For conversion factors, consult the last page of this assignment

1. (a) The average volume of an adult human eyeball is 5.5 cm^3. What is its volume expressed in in^3 (cubic inches)?

(b) The average surface area of adult human skin is about 1.8 m^2. Convert it to cm^2 and to mm^2.

(c) The average density of a human jaw bone is 1.069 oz/in^3 (ounces per cubic inch). Convert this quantity into g/cm^3.

(d) Average blood flow through an adult human brain is between 750 and 1000 ml/min (millilitres per minute). Express this range in l/s (litres per second).

(e) The average speed of a garden snail is 0.03 mph (miles per hour). Convert it into m/s (metres per second).

(f) A large banana slug covers 6.5 inches in 120 minutes. What is its speed in km/h ?

Continued on next page

2. (a) Average time between (two consecutive) blinks of an eye is about 2.8 seconds. How many times does an eye blink in one day?

(b) The density of touch receptors on a human fingertip is about 2,500 per cm^2. How many touch receptors are there on a patch of a fingertip whose area is 50 mm^2?

(c) The average rate of production of cerebrospinal fluid (CSF) in an adult is 0.35 ml/min (millilitres per minute). What is the amount (in litres) of CSF produced in 4 hours?

Continued on next page

3. When we convert from one set of units to another, numbers change. For instance, 10 kg is not equal to 10 lb, but to $10 \cdot 2.2046 = 22.046$ lb. Or, 2 hours is not 2 minutes, but 2 hours = 120 minutes. However, there are situations when we would like to obtain **the same number no matter what units are used**, such as in calculating the body mass index. If we wish to do that, we have to adjust the corresponding formula.

Go through the following two exercises to practice this concept. Then go back to Assignment 6, and work on question 5.

(a) Consider the mass-time index $I = m \cdot T$, where m is mass in kg (kilograms) and T is time in min (minutes).
(i) Find the value of I when $m = 4$ kg and $T = 30$ min.

(ii) Use the same formula $I = m \cdot T$ with the mass converted to g (grams) and the time T (still) in minutes. Clearly, the numeric values of I in (i) and (ii) do not agree.

(iii) Now adjust the formula for I (call the new index J) so that when you substitute the mass in grams and the time in minutes into J you get the same value as when you substitute the mass in kilograms and the time in minutes into I.

Continued on next page

(b) Consider the mass-time index $I = m \cdot T$, where m is mass in kg (kilograms) and T is time in min (minutes).
(i) Find the value of I when $m = 4$ kg and $T = 30$ min.

(ii) Use the same formula $I = m \cdot T$ with the mass converted to g (grams) and the time T in h (hours). Clearly, the numeric values of I in (i) and (ii) do not agree.

(iii) Now adjust the formula for I (call the new index J) so that when you substitute the mass in grams and the time in hours into J you get the same value as when you substitute the mass in kilograms and the time in minutes into I.

Continued on next page

(c) (i) Compute the body mass index BMI=m/h^2 for a person of mass 62 kg and height 1.5 m.

(ii) Find a formula for the body mass index (call it BMJ) where the mass is in grams and the height is in metres. Check it for the person from (i).

(iii) Find a formula for the body mass index (call it BMK) where the mass is in kilograms and the height is in centimetres. Check it for the person from (i).

IMPORTANT UNITS

NEED TO REMEMBER

Time: hour (h), minute (min), second (s)
Length: metre (m), centimetre (cm), milimetre (mm), foot (ft), inch (in), yard (yd), mile (mile)
Mass: kilogram (kg), gram (g), ounce (oz), pound (lb)

Time
1 h = 3,600 s = 60 min
1 year = 365.24 days
1 day = 24 h

Length
1 m = 100 cm = 1000 mm
1km = 1000 m
1 ft = 12 in

Mass
1 kg = 1000 g
1 lb = 16 oz

Angle
180 degrees = π radians

WILL BE GIVEN WHEN NEEDED (NO NEED TO MEMORIZE)

Length
1 m = 3.2808 ft
1 ft = 0.3048 m
1 in = 2.54 cm
1 cm = 0.3937 in
1 km = 0.6214 mile
1 mile = 1.6093 km
1 yd = 36 in

Mass
1 kg = 2.2046 lb
1 lb = 0.4536 kg
1 g = 0.0353 oz
1 oz = 28.35 g

Chapter 2 (geese) 1 (elephants) concepts

1. (a) What do we mean when we say that a quantity M is inversely proportional to a quantity P?

(b) What is a linear function?

(c) Is $f(x) = 3x + 1 + x^{-1}$ a linear function?

(d) In which case does a linear relationship $y = ax + b$ represent a proportional relationship between x and y?

(e) What is a difference between a linear relationship $y = ax + b$ (where $b \neq 0$) and a proportional relationship?

(f) Give an example of a linear function (as a table of values) which does not represent a proportional relationship.

Continued on next page

2. (a) Consider the function $f(x) = x^r$, where $x > 0$. For which r is f a decreasing function?

(b) In the same coordinate system, sketch the graphs of $f(x) = x^3$ and $g(x) = x^4$ for $x \geq 0$. Which function increases faster?

(c) In the same coordinate system, sketch the graphs of $f(x) = x^{1/2}$ and $g(x) = x^{1/3}$ for $x \geq 0$. Which function increases faster?

(d) In the same coordinate system, sketch the graphs of $f(x) = x^{-3}$ and $g(x) = x^{-4}$ for $x > 0$. Which function decreases faster?

Continued on next page

3. Read Example 2.1.10 (geese) 1.1.11 (elephants). Use the same idea to build a linear model for the population of Canada based on the data for 1986 and 1996 (look at Table 1.4.1, page 29 (geese) Table 0.3.1, page 26 (elephants)). Compare with the result of Example 2.1.10 (geese) 1.1.11 (elephants).

4. In Example 2.1.9 (geese) 1.1.10 (elephants) we derived the formula that converts degrees Celsius to degrees Fahrenheit. Modify the derivation to obtain the formula that converts degrees Fahrenheit to degrees Celsius.

Continued on next page

5. If we wish to calculate the body mass index of a person whose weight is in pounds and height is in inches, we can no longer use BMI $= m/h^2$. Find the formula for the BMI in that situation.

6. Student A, of height h metres and mass m kilograms, has the body mass index of BMI $= m/h^2$. Student B has the same mass as student A, but is 5% taller than student A. Student C is of the same height as student A, but their mass is 5% less than the mass of student A. Which of the two students, B or C, has lower BMI?

Continued on next page

7. Read Example 2.1.12 (geese) 1.1.13 (elephants). Assume that the blood circulation time is proportional to the *third* root of the body mass (instead of the fourth root, as in the example). Based on this modified model, answer the following questions:
(a) If the body mass doubles, how does the blood circulation time change?

(b) Find a formula for the blood circulation time, knowing that a 5400 kg elephants has a blood circulation time of 152 seconds.

(c) Using your formula from (b), calculate the blood circulation time for a mouse of mass 0.1 kg and a sperm whale of mass 38,000 kg, and compare with the answers at the end of Example 2.1.12 (geese) 1.1.13 (elephants).

Continued on next page

8. (a) Read Example 2.1.14 (geese) 1.1.15 (elephants) and, in one sentence, state the relationship between the surface area and the volume of an animal.

(b) Summarize Example 2.1.16 (geese) 1.1.17 (elephants) in no more than three sentences: say what it is about, and what the main point is.

(c) Read Example 2.2.13 (geese) 1.2.13 (elephants). What does it say about limitations of radiocarbon dating?

(d) Exercise 69 on page 91 (geese) 63 on page 75 (elephants).

Continued on next page

9. Read Example 2.2.16 (geese) 1.2.16 (elephants). Draw a semilog graph of $S(t) = 24e^{1.8t}$ against t. In the same coordinate system, draw a semilog graph of $T(t) = 24e^{-0.8t}$.

10. Exercise 66 on page 109 (geese) 54 on page 89 (elephants).

Continued on next page

11. Identify the minimum, maximum, average, amplitude, period and phase for the following oscillations.

(a) $y = \sin(4t + \pi)$

(b) $y = \cos(t/2) + 5$

(c) $y = -2\sin(3t) + 4$

12. A population of salmon changes periodically with a period of 12 months. In January, it reaches its minimum of 2 million, and in July it reaches its maximum of 4.6 million. Find a formula that describes how the salmon population changes with time.

THE END

Section 2.2 (geese) 1.2 (elephants): Semilog and double-log plots

1. The amount of carbon-14 (^{14}C) left t years after the death of an organism is given by

$$Q(t) = 6 \cdot 10^{10} e^{-0.000122t}$$

where $Q(t)$ counts the number of ^{14}C atoms.

(a) Sketch a semilog graph of $Q(t)$, using $\ln$.

(b) Sketch a semilog graph of $Q(t)$, using $\log_{10}$.

Continued on next page

2. Consider the population of bacteria growing according to $P(t) = P(0)e^{kt}$. Assume that $P(0) = 100$ and $k = 0.56$.

(a) Find the doubling time t_d of $P(t)$.

(b) Sketch the graph of $P(t)$ is the usual coordinate system (i.e., $P(t)$ vs. t) and label the points which correspond to $t = 0$, $t = t_d$, $t = 2t_d$, $t = 3t_d$, and $t = 4t_d$.

(c) In your graph in (b), indicate the changes in the values of $P(t)$ as t changes from 0 to t_d, then from t_d to $2t_d$, and so on. What pattern of growth do these changes show?

Continued on next page

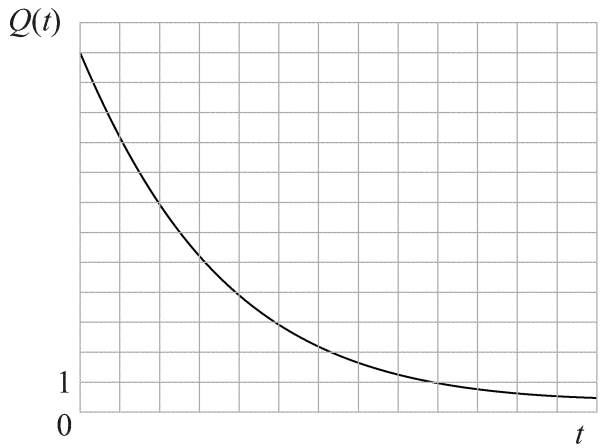
(d) Sketch the semilog graph of $P(t)$ (i.e., $\ln P(t)$ vs. t) and label the points which correspond to $t = 0$, $t = t_d$, $t = 2t_d$, $t = 3t_d$, and $t = 4t_d$.

(e) In your graph in (d), indicate the changes in the values of $P(t)$ as t changes from 0 to t_d, then from t_d to $2t_d$, and so on. What pattern of growth do these changes show?

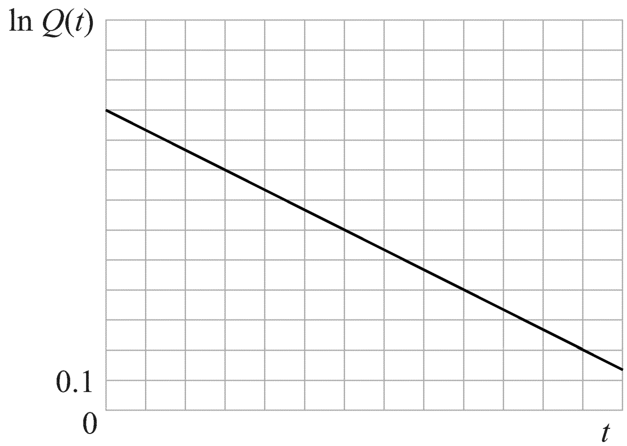
(f) Repeat (d) and (e) the semilog graph of $P(t)$ where $\log_{10}$ is used instead of $\ln$.

Continued on next page

3. (a) The graph below shows an exponentially decreasing quantity $Q(t)$. Identify the point on the t-axis which represents the half-life t_h of $Q(t)$. Identify the point on the t-axis where $Q(t)$ decreases 16-fold.



(b) The semilog graph below shows an exponentially decreasing quantity $Q(t)$. Identify the point on the t axis which represents the half-life t_h of $Q(t)$.



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4. Consider the formula $h = 241B^{-0.25}$ for the dependence of the heartbeat frequency h on body mass B of a mammal. B is measured in kilograms and units of h are 1/min.

(a) Use ln to sketch a double-log plot of h. Label the axes and indicate a reasonable domain.

(b) Use $\log_{10}$ to sketch a double-log plot of h. Label the axes and indicate a reasonable domain.

Continued on next page

5. The resistance R of the flow of blood through a blood vessel (assumed to have the shape of a cylindrical tube) is given by

$$R = \frac{Kl(\gamma+1)^2}{d^4}$$

where l is the length of the vessel, d is its diameter and $\gamma \geq 0$ is the curvature. The positive constant K represents the viscosity of the blood (viscosity is a measure of the resistance of fluid to stress; water has low viscosity, honey has high viscosity).
In this exercise we view R as a function of d.

(a) Sketch the semilog graph of R (use ln). Label the axes and identify a reasonable domain for d.

(b) Sketch the double-log graph of R (use ln). Label the axes and identify a reasonable domain for d.

THE END

Sections 4.1, 4.2 (geese) 3.1, 3.2 (elephants): Rates of change and limits

1. Find the average rate of change between the base point t_0 and times $t_0 + \Delta t$, where $\Delta t = 0.5$, $\Delta t = 0.1$ and $\Delta t = 0.01$. Also, find the equation of the secant line connecting the given base point t_0 and time $t_0 + 0.1$.

(a) $f(t) = t^2 - 1$, base point $t_0 = 1.5$

(b) $f(t) = e^{3t}$, base point $t_0 = 0$.

Continued on next page

2. A population changes according to the formula $P(t) = 1.37^t$, where t is time in days.
(a) Find the average change between times 4 and 5.

(b) Find the average change between times 4 and 4.5.

(c) Find the average change between times 4 and 4.1.

3. A population changes according to the formula $P(t) = 5.34e^{0.7t}$, where t is time in years.
(a) Find the average change between times 2 and 3.

(b) Find the average change between times 2 and 2.1.

Continued on next page

4. Looking at the graph of the function involved, determine the following limits.

(a) $\lim_{x \to 0} \frac{1}{x-1}$

(b) $\lim_{x \to 0} \frac{1}{x^2}$

(c) $\lim_{x \to 1} \frac{x^2-1}{x-1}$

(d) $\lim_{x \to 2^+} \sqrt{x-2}$

Continued on next page

5. Looking at the graph of the function involved, determine the following limits.

(a) $\lim_{x \to 0^+} \log_{10} x$

(b) $\lim_{x \to 0} e^{-1.2x}$

(c) $\lim_{x \to -\pi} 2\cos(x - \pi)$

(d) $\lim_{x \to \pi/2} \tan x$

Continued on next page

6. Compute the following limits.

(a) $\lim\limits_{x\to 0} \dfrac{x^2-4}{x+2}$

(b) $\lim\limits_{x\to -2} \dfrac{x^2-4}{x+2}$

(c) $\lim\limits_{t\to 3} \dfrac{t^3-27}{t+3}$

(d) $\lim\limits_{t\to 3} \dfrac{t^3-27}{t-3}$

(e) $\lim\limits_{t\to 2} \dfrac{\frac{1}{2}-\frac{1}{t}}{t-2}$

Continued on next page

7. Let $h(x) = \sqrt{\dfrac{4.7}{x-4}}$.

(a) Find the domain of $h(x)$.

(b) Compute the limit of $h(x)$ as x approaches 4 from the right.

8. (a) Sketch the graph of the function

$$f(x) = \begin{cases} 2x-2 & \text{if} \quad x < 1 \\ 0 & \text{if} \quad 1 \le x \le 2 \\ 1/x & \text{if} \quad x > 2 \end{cases}$$

(b) From (a), determine (if they exist) $\lim_{x\to 1} f(x)$ and $\lim_{x\to 2} f(x)$.

Continued on next page

9. Determine the following limits.

(a) $\displaystyle\lim_{x\to 2^+} \frac{1}{x-2}$

(b) $\displaystyle\lim_{x\to 2^-} \frac{1}{x-2}$

(c) $\displaystyle\lim_{y\to 2} \frac{\ln y}{y^2-3}$

(d) $\displaystyle\lim_{y\to 0} \sin e^{-y^2}$

(e) $\displaystyle\lim_{t\to -1} \frac{\tan(\pi t)}{t-1}$

THE END

Math 1LS3	Assignment 9

Section 4.3 (geese) 3.3 (elephants): Limits involving infinity

1. Find the following limits.

(a) $\lim_{x\to\infty} \left(x^3 - 2x + 4.3\right)$

(b) $\lim_{x\to-\infty} \left(x^3 - 2x + 4.3\right)$

(c) $\lim_{x\to\infty} \left(x^2 - 2.2x - 10000000\right)$

(d) $\lim_{x\to-\infty} \left(x^2 - 2.2x - 10000000\right)$

(e) $\lim_{x\to\infty} 0.65^x$

(f) $\lim_{x\to-\infty} 0.65^x$

(g) $\lim_{x\to\infty} e^{3x}$

(h) $\lim_{x\to-\infty} e^{-0.99x}$

Continued on next page

2. Find the following limits.

(a) $\lim_{x\to\infty} 3.76^x$

(b) $\lim_{x\to\infty} 3.76^{-x}$

(c) $\lim_{x\to\infty} e^{-x^2}$

(d) $\lim_{x\to-\infty} e^{-x^2}$

(e) $\lim_{x\to\infty} \ln(x-100)$

(f) $\lim_{x\to\infty} \ln(x/10)$

(g) $\lim_{x\to\infty} \sqrt{x^2+4500}$

(h) $\lim_{x\to-\infty} \sqrt{1-x^3}$

Continued on next page

3. Find the following limits (or say that the limit does not exist)

(a) $\lim_{x\to\infty} 1^x$

(b) $\lim_{x\to\infty} \ln(1-x^5)$

(c) $\lim_{x\to-\infty} \ln(1-x^5)$

(d) $\lim_{x\to\infty} \sqrt{x/10}$

(e) $\lim_{x\to\infty} \sqrt{10000-x}$

(f) $\lim_{x\to\infty} e^{x^2-x-4}$

(g) $\lim_{x\to\infty} e^{-x^2-x+44}$

Continued on next page

4. Graph each pair of functions in the same coordinate system and identify the one that approaches ∞ faster as x approaches ∞.

(a) $1000x^2$ and $0.03x^3$

(b) x^2 and $0.1e^x$

(c) x^2 and $e^{0.1x}$

(d) x and $\sqrt{x}$

Continued on next page

(e) x and $\ln x$

(f) $\sqrt{x}$ and $\ln x$

(g) $x^{1/3}$ and $\ln x$

(h) $10e^x$ and $e^{1.2x}$

Continued on next page

5. Find the following limits.

(a) $\lim_{x\to\infty} \dfrac{2-3x^3}{1-x^2+2x^3}$

(b) $\lim_{x\to\infty} \dfrac{1.4x^2}{1-0.6x^2+24x^3}$

(c) $\lim_{x\to\infty} \dfrac{0.001x^4}{1+20000x^3}$

(d) $\lim_{x\to\infty} \Big(\ln(2x^4+1)-\ln(x^2-13)\Big)$

Continued on next page

6. Graph each pair of functions in the same coordinate system and identify the one that approaches 0 faster as x approaches ∞.

(a) x^{-2} and x^{-1}

(b) x^{-2} and $0.1x^{-1}$

(c) x^{-10} and e^{-x}

(d) e^{-2x} and e^{-x}

Continued on next page

7. Find the following limits (or else say that they do not exist)

(a) $\lim_{x\to\infty} \dfrac{e^x - e^{-x}}{2e^{-x} + e^x}$ [hint: divide numerator and denominator by e^x]

(b) $\lim_{x\to-\infty} \dfrac{e^x - e^{-x}}{2e^{-x} + e^x}$ [hint: divide numerator and denominator by ... (you figure it out)]

THE END

Math 1LS3 Assignment 10

Sections 4.2–4.4 (geese) 3.2–3.4 (elephants): Concepts of limits and continuity

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1. (a) Define: the limit of a function $f(x)$ as x approaches a is L. Do not copy the definition from the textbook; instead, try to rephrase in your own words.

(b) What is the point of Example 4.2.3 (geese) 3.2.3 (elephants)?

(c) Read Example 4.2.7 (geese) 3.2.6 (elephants). What is the limit of $f(x)$ as $x \to 1$?

(d) In a sentence form, summarize the properties of limits given in Theorem 4.2.1 (geese) 3.1 (elephants).

(e) What does the direct substitution rule say?

Continued on next page

2. (a) Explain what do we mean when we say that "$f(x)$ approaches ∞ as $x \to a$".

(b) In words, explain why $\lim_{x\to 0} \frac{1}{x}$ does not exist.

(c) Define: vertical asymptote.

(d) Define: horizontal asymptote.

(e) Which function approaches ∞ faster as x approaches ∞: x^3 or $x^{3.7}$?

(f) Which function approaches 0 faster as x approaches ∞: x^{-3} or $x^{-3.7}$?

Continued on next page

3. (a) Explain what "$f(x)$ is continuous on an interval" means.

(b) In one sentence, summarize Theorem 4.4.1 (geese) 3.4 (elephants).

(c) In one sentence, summarize Theorem 4.4.2 (geese) 3.5 (elephants).

(d) Give one example of a discontinuous function for each case listed in Table 4.4.1 (geese) 3.4.1 (elephants).

Continued on next page

4. (a) Write an equation that expresses the fact that a function $f(x)$ is continuous at $x = 2$.

(b) Sketch the graph of a function that satisfies $\lim_{x \to 2} f(x) = 3$, and that is not continuous at $x = 2$.

(c) Using the definition in (a), show that the function

$$f(x) = \begin{cases} x^2 - 1 & \text{if} \quad x < 2 \\ 2.5 + 1/x & \text{if} \quad x \geq 2 \end{cases}$$

is continuous at $x = 2$.

Continued on next page

5. Find value(s) of x for which the function

$$f(x) = \begin{cases} 1 - \dfrac{1}{x} & \text{if} \quad x < 2 \\ \dfrac{1}{x-1} & \text{if} \quad x \geq 2 \end{cases}$$

is not continuous. Justify your answer.

6. Find all x for which the function $f(x) = \ln(e^x - 1)$ is continuous. Justify your answer.

THE END

Math 1LS3	**Assignment 11**

Sections 4.5, 5,1, 5.2 (geese) 3.5, 4,1 4.2 (elephants): Derivatives

1. Using the definition, find $f'(x)$.

(a) $f(x) = 3x - 5$

(b) $f(x) = 1 - x^2$

(c) $f(x) = \sqrt{x}$

Continued on next page

(d) $f(x) = \dfrac{2}{x}$

(e) $f(x) = \dfrac{1}{\sqrt{x}}$

(f) $f(x) = \dfrac{x}{x+1}$

Continued on next page

Find the derivatives of the following functions.

2. (a) $y = x + 1 + x^{-1}$

(b) $y = \dfrac{\sqrt{12}}{x} + \dfrac{x}{\sqrt{12}}$

(c) $y = \sqrt{3x} + \sqrt{3}\,x$

3. Find $f'(2)$ if $f(x) = \dfrac{x^2 - 2x^3 - 4}{12 + x^2}$.

Continued on next page

4. (a) What is a differentiable function?

(b) What is a corner? Is a function differentiable at a corner? Why, or why not?

(c) What is a critical point (critical number)?

(d) If $m(t)$ represents the number of monkeys at time t, what are the units of $m'(t)$?

5. Exercise 9, page 254 (geese) 9, page 226 (elephants) in your textbook.

Continued on next page

6. Find the equation of the line tangent to the graph of the function $f(x) = x - 2\sqrt{x}$ at the point $x = 1$.

7. (a) Sketch the graph of a function defined for all x which is positive for all x, and whose derivative is also positive for all x.

(b) Sketch the graph of a function defined for all x which is negative for all x, but whose derivative is positive for all x.

Continued on next page

8. True/false questions. Decide whether the statements in questions (a) and (b) are true or false (circle your choice).You must justify your answer.
(a) If a function f is positive (i.e., $f(x) > 0$ for all x), then its derivative f' is positive.

TRUE FALSE

(b) If a function f is increasing for all x, then its derivative f' is increasing for all x.

TRUE FALSE

9. The curve $y = x^3 - x^2 - x$ has two horizontal tangent lines. Find the x-coordinates of the points of tangency.

Continued on next page

10. Let

$$f(x) = \begin{cases} 1 & \text{if} \quad x < -1 \\ 2 - x^2 & \text{if} \quad -1 \leq x \leq 1 \\ x & \text{if} \quad x > 1 \end{cases}$$

Sketch the graph of $f(x)$. Looking at the graph of $f(x)$, sketch the graph of $f'(x)$.

11. Sketch the graph of $f(x) = |x + 2|$. Find all values for x where the function $f(x)$ is not differentiable.

THE END

Math 1LS3	Assignment 12

Sections 5.1, 5.3, 5.5 (geese) 4.3, 4.4 (elephants): Techniques of differentiation

1. Find the derivatives of the following functions (a and b are constants).

(a) $y = ax + be^x$

(b) $y = axe^{bx}$

(c) $y = \ln(ax + b)$

(d) $y = \ln(ax) + \ln(bx)$

(e) $y = ax\ln(bx)$

(f) $y = \ln(ax)\ln(bx)$

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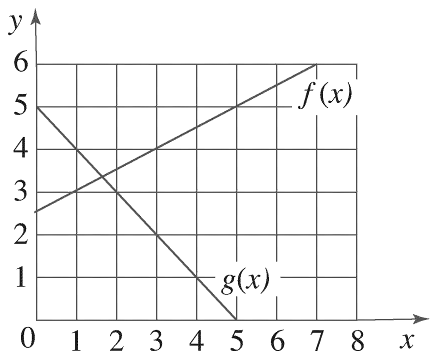
2. The graphs of the functions $f(x)$ and $g(x)$ are given below. Find $(g \circ f)(1)$ and $(g \circ f)'(1)$



3. Find an equation of the line tangent to the graph of the function $y = \dfrac{e^x + 1}{x}$ at the point where $x = 1$.

Continued on next page

4. Find the derivatives of the following functions. Do not simplify your answers.

(a) $y = (x^3 - x - 1)^6$

(b) $y = \sqrt{x^4 - 14x} + (\sqrt{x} + 1)^2$

(c) $y = xe^{x^2+x}$

(d) $y = e^x + x^e + e^e$

(e) $y = e^{\sqrt{x}} + \sqrt{e^x}$

Continued on next page

5. (a) Find the equation of the line tangent to the graph of the function $f(x) = \ln(x-3)$ at the point where $x = 4$.

(b) In the same coordinate system, sketch both the function $f(x)$ and its tangent from (a).

6. (a) Find the domain of the function $g(x) = \dfrac{1}{\ln x + 1}$.

(b) For the function in (a), find $g(1)$ and $g'(1)$.

Continued on next page

7. Find the derivatives of the following functions.

(a) $y = \dfrac{a\ln x + b}{c\ln x + d}$ (a, b, c, and d are constants)

(b) $f(x) = x\ln(1 + e^x)$

(c) $f(x) = \ln\sqrt{x} + \ln(\sqrt{x} + 1)$

(d) $f(x) = \ln\left(\dfrac{x^2 e^x}{x^3 + 1}\right)$ (hint: simplify using properties of ln first)

Continued on next page

8. Using implicit differentiation, find y'.

(a) $e^{xy^2} - x - y^2 = 0$

(b) $\sin x + \sin x \sin y = \cos(xy)$

(c) $(x^2 + y^2)^{-1} = x$

THE END

Section 5.4 (geese) 4.5 (elephants): Derivatives of trig and inverse trig functions

1. Compute the derivatives of the following functions. Do not simplify your answers.

(a) $f(x) = \sin 2x + 2 \sin x$

(b) $y = \cos^2 x + \cos(x^2)$

(c) $f(x) = \tan(x^2 - 5x + 7)$

(d) $f(x) = e^{\tan(x^2 - 5x + 7)}$

(e) $y = x^2 \sin(1/x)$

Continued on next page

2. Compute the derivatives of the following functions. Do not simplify your answers.
(a) $z = \arctan x + \arctan(x^2)$

(b) $z = \arcsin x + (\arcsin x)^2$

(c) $z = \arcsin(x^3 - 11x + 4)$

(d) $z = \sin(\arctan x) + \tan(\arcsin x)$.

Continued on next page

3. For what value(s) of x does the graph of $y = x - 2\sin x$ have a horizontal tangent?

4. Starting from $\tan x = \dfrac{\sin x}{\cos x}$, derive the formula $(\tan x)' = \sec^2 x$.

5. Starting from $\sec x = \dfrac{1}{\cos x}$, derive the formula $(\sec x)' = \sec x \tan x$.

Continued on next page

6. Starting from $\sin(\arcsin x) = x$, find the derivative of $\arcsin x$. Write the answer without trigonometric functions.

7. Starting from $\tan(\arctan x) = x$, find the derivative of $\arctan x$. Write the answer without trigonometric functions.

THE END

Sections 5.6, 5.7 (geese) 4.6, 4.7 (elephants): Graphs and approximations of functions

1. (a) State the connection between the second derivative of a function and concavity. What is an inflection point?

(b) If $f''(x) = 0$, does it mean that x is an inflection point? Explain.

(c) Read Example 5.6.3 (geese) 4.6.2 (elephants). Then, using the first and the second derivatives, analyze and sketch the graph of $f(x) = -2x^2 - 3x + 7$.

Continued on next page

2. Read Example 5.6.10 (geese) 4.6.10 (elephants). Then, using the first and the second derivatives, analyze and sketch the graph of $f(x) = x^3 - 9x$.

3. Exercise 2 on page 327 (geese) 2 on page 290 (elephants) in your textbook.

Continued on next page

4. Let $g(z) = z + \frac{1}{z}$, where $z > 0$. Find the first and the second derivatives of $g(z)$ and use them to sketch the graph of $g(z)$.

5. (a) Find $f^{(100)}(x)$ for $f(x) = \cos x$.

(b) Find $f^{(66)}(x)$ for $f(x) = e^{-2x}$.

Continued on next page

6. (a) What is a linear approximation?

(b) Find a linear approximation of $f(x) = e^{-3x}$ at $x = 0$.

(c) Explain how to obtain a secant line approximation.

(d) Find a secant line approximation of $f(x) = e^{-3x}$ based on $x = 0$ and $x = 1$.

(e) What is a quadratic approximation? Write down a formula for the Taylor polynomial of degree 2 at $x = 0$.

(f) Find a quadratic approximation of $f(x) = e^{-3x}$ at $x = 0$.

Continued on next page

7. (a) Explain in words how to estimate $\ln 0.94$ using a tangent line. [Hint: read Example 5.7.2 (geese) 4.7.2 (elephants).] Find an estimate for $\ln 0.94$.

(b) Read Example 5.7.7 (geese) 4.7.7 (elephants) and improve the estimate you obtained in (a).

Continued on next page

8. (a) Exercise 10, page 340 (geese) 10, page 303 (elephants).

(b) Exercise 16, page 340 (geese) 16, page 303 (elephants).

THE END

Section 5.7 (geese) 4.7 (elephants)

1. (a) Compute Taylor polynomials $T_1(x)$, $T_2(x)$ and $T_3(x)$ for the function $f(x) = \sqrt{x}$ near (or at) $a = 4$.

(b) Using the polynomials from (a), find estimates for $\sqrt{4.5}$ and compare with the calculator value $\sqrt{4.5} = 2.121320344$.

Continued on next page

2. Consider the function $f(x) = x^2 + 1/x$. Approximate the value $f(1.4)$ using each approach below.

(a) Tangent line approximation $L_1(x)$ (i.e. tangent line with the basepoint at $a = 1$).

(b) Tangent line approximation $L_2(x)$ (i.e. tangent line with the basepoint at $a = 2$).

(c) Secant line approximation $S(x)$ defined by the points $x = 1$ and $x = 2$.

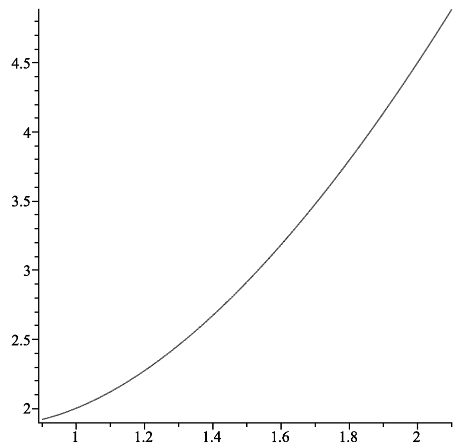
Continued on next page

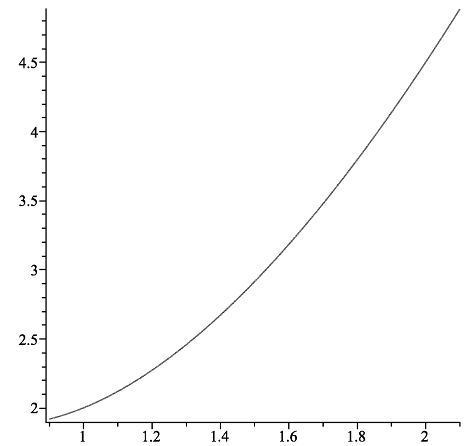
(d) Taylor polynomial approximation $T_2(x)$ based at at $a = 1$.

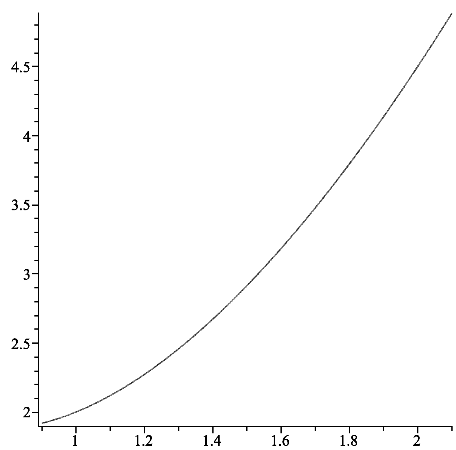
(e) Taylor polynomial approximation $T_2(x)$ based at at $a = 2$.

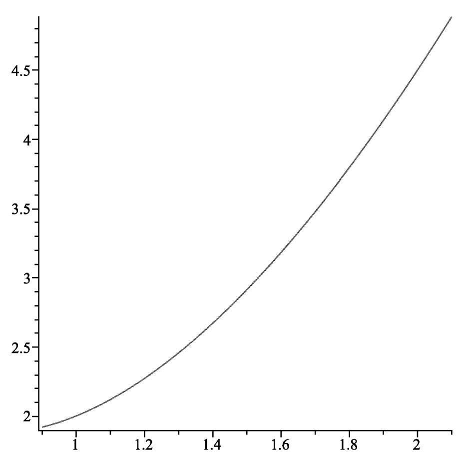
Continued on next page

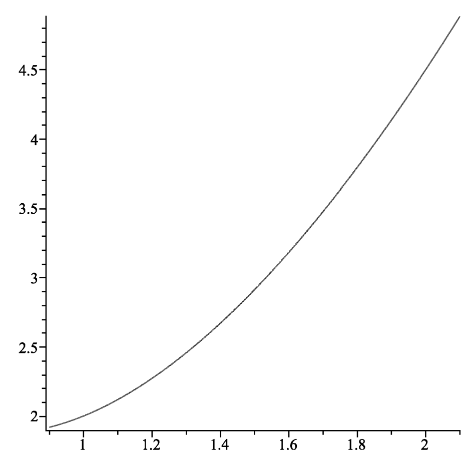
(f) Given is the graph of $f(x) = x^2 + 1/x$. Sketch each approximation (a)-(e).











THE END

Section 6.1 (geese) 5.1 (elephants): Extreme values of a function

1. (a) What is a relative maximum of a function?

(b) What is the difference between a relative maximum and an absolute maximum?

(c) What does Fermat's theorem claim?

(d) Does Fermat's theorem imply that a function for which $f'(c) = 0$ has a local maximum or a local minimum at c?

(e) In your words, explain what the first derivative test says.

(f) Give an example of a function and a critical point to which you can apply the first derivative test, but not the second derivative test.

Continued on next page

2. (a) State the Extreme Value Theorem.

(b) Give an example (a formula, or a graph) of a continuous function defined on $(0,4)$ which has a global maximum but no global minimum.

3. Read the first part of Example 6.1.18 (geese) 5.1.14 (elephants), up to the paragraph "Now let us look ...". How does the resistance R change if the diameter of the tube (i.e., the blood vessel) gets reduced by 20%?

Continued on next page

4. Find all critical points of the following functions.
(a) $f(x) = x^3 - x + 2$

(b) $f(x) = x^3 + x + 2$

(c) $f(x) = x \ln x$

Continued on next page

(d) $f(x) = \sin x + \cos x$

(e) $f(x) = |x + 2|$ (hint: sketch the graph)

(f) $f(x) = xe^{3x}$

Continued on next page

5. Find the absolute extreme values of the function $f(x) = x^4 - 4x + 3$ on the interval $[0, 2]$.

6. Find the absolute maximum and the absolute minimum of the function $f(x) = \dfrac{\ln x}{x}$ on the interval $[1, 3]$.

Continued on next page

7. Is it true that every function defined on a closed interval $[a, b]$ has an absolute maximum and an absolute minimum? Explain your answer.

8. It is known that $f'(2) = 0$. Which of the following statements is/are true?
(I) f is continuous at $x = 2$
(II) f has a relative extreme value at $x = 2$
(III) f has a horizontal tangent at $x = 2$

9. Give an example (sketch a graph, or write down a formula) of a continuous function $f(x)$ such that $f'(1) = 0$, but f does not have an extreme value at $x = 1$.

Continued on next page

10. Exercise 64, page 364 (geese) 48, page 325 (elephants).

11. Sketch a graph of a continuous function on an interval $[a, b]$ which
(a) has both global extreme values at its endpoints and no critical points inside the interval.

Continued on next page

(b) has both global extreme values at its endpoints and two critical points inside the interval.

(c) has global maximum at both endpoints and global minimum within the given interval.

(d) has global maximum at both endpoints, global minimum within the given interval, and a total of three critical points inside the interval.

THE END

Math 1LS3	**Assignment 17**

Section 6.4 (geese) 5.3 (elephants): Leading behaviour

1. Identify the leading behaviour of the following functions at 0 and at ∞.

(a) $f(x) = 3 + 2x$

(b) $f(x) = 7 - 2x^2$

(c) $f(x) = x - 2e^x$

(d) $f(x) = 3x - 1000x^3 + x^4$

(e) $f(x) = x + 2x^2 + x^{-1}$

(f) $f(x) = x^{-1} + 2x^{-3} + 4x^{-5}$

Continued on next page

2. Decide which of the two functions approaches ∞ faster as x approaches ∞ and check your answer using L'Hôpital's rule or otherwise.

(a) $f(x) = 3 + 200x$ and $g(x) = 0.01x^2 - 100$

(b) $f(x) = 2\sqrt{x}$ and $g(x) = 25 \ln x$

(c) $f(x) = x + x^{10}$ and $g(x) = e^x$

(d) $f(x) = x^{0.4}$ and $g(x) = \sqrt{x}$

Continued on next page

3. Decide which of the two functions approaches zero faster, and check using L'Hôpital's rule or otherwise.

(a) $f(x) = 200x$ and $g(x) = 0.01x^2$, as x approaches 0

(b) $f(x) = x^{-2}$ and $g(x) = e^{-3x}$, as x approaches ∞

(c) $f(x) = x^{-2}$ and $g(x) = 1000x^{-1}$, as x approaches ∞

(d) $f(x) = x^2$ and $g(x) = 0.1x^3$, as x approaches 0

Continued on next page

4. For each of the following functions
(i) find the leading behaviour at both 0 and ∞.
(ii) find the limit at both 0 and ∞.
(iii) sketch the graph.

(a) $f(x) = \dfrac{x^2}{2x+1}$

(b) $f(x) = \dfrac{x^2}{2x^2+1}$

(c) $f(x) = \dfrac{1+x+x^2}{2x+3}$

Continued on next page

(d) $f(x) = \dfrac{e^x}{2e^x + x}$

THE END

Math 1LS3	Assignment 18

L'Hopital's Rule, Section 6.4 (geese) 5.3 (elephants)

1. Which of the following limits are indeterminate forms? (Just say 'yes' or 'no,' do not compute the limits)

(a) $\lim_{x\to 0} \frac{\sin x}{\tan x}$

(b) $\lim_{x\to 0} \frac{\cos x}{\tan x}$

(c) $\lim_{x\to 0} x^x$

(d) $\lim_{x\to \infty} x^x$

(e) $\lim_{x\to 1} x^x$

(f) $\lim_{x\to \infty} (x - \ln x)$

(g) $\lim_{x\to \infty} (x + \ln x)$

Continued on next page

2. Find the following limits.

(a) $\lim_{x \to 0^+} \frac{3 \ln x}{x^2}$

(b) $\lim_{x \to 0} \frac{\arcsin x}{x}$

(c) $\lim_{x \to \infty} \left(\sqrt{x^2 - x} - x \right)$

Continued on next page

3. Find the following limits.

(a) $\lim_{x\to\infty} \dfrac{3x}{\ln(2+e^x)}$

(b) $\lim_{x\to 0^+} (1-3x)^{1/x}$

(c) $\lim_{x\to 0} \dfrac{e^{x^2}-1-x^2}{x^4}$

Continued on next page

4. Compute the following limits.

(a) $\lim_{x \to \infty} x^2 e^{-1.3x}$

(b) $\lim_{x \to 0^+} (4x)^{5x}$

(c) $\lim_{x \to \infty} \frac{\ln(\ln x)}{x^2}$

Continued on next page

(d) $\lim_{x \to \pi/2^-} (\sec x - \tan x)$

(e) $\lim_{x \to \infty} \dfrac{e^x + 4x}{x^2 + 3e^x}$

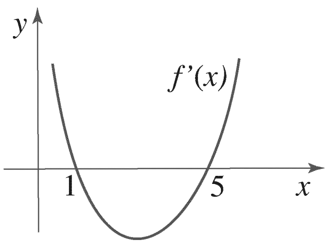
(f) $\lim_{c \to \infty} \dfrac{4c}{2 + \ln(1 + c)}$

THE END

Math 1LS3	Assignment 19

Sections 7.1, 7.2 (geese) 6.1, 6.2 (elephants): Differential equations and antiderivatives

1. The graph of the derivative f' of a function f is given below.



(a) On what intervals is f increasing? Decreasing?

(b) At what values of x does f have a maximum?

(c) Sketch the graph of f if $f(0) = 1$.

Continued on next page

2. Find the most general antiderivative of each of the following functions.

(a) $f(x) = \dfrac{1}{\sqrt{x}} + \dfrac{1}{x} + \dfrac{1}{\sqrt{x^3}}$

(b) $y = \pi^x + x^\pi + \pi^\pi$

(c) $f(x) = 3\sin x + 4\cos x + 5$

(d) $f(x) = e^x + x^{-2}$

(e) $y = \sec x \tan x - 1$

(f) $f(x) = \dfrac{x - 2x^2 + 1}{\sqrt{x}}$

Continued on next page

3. Find the following indefinite integrals.

(a) $\int (\sec^2 x + 3\cos x)\, dx$

(b) $\int \left(\frac{1.4}{5\sqrt{x}} + \frac{2}{\sqrt[5]{x}} \right)\, dx$

(c) $\int \frac{6}{1+x^2}\, dx$

4. Exercise 13, page 471 (geese) 13, page 416 (elephants).

Continued on next page

5. (a) Prove that $y(x) = e^{1-\sqrt{1+x^2}} - 1$ is a solution of the initial value problem

$$y'(x) = -x(1+y)(1+x^2)^{-1/2}, \quad y(0) = 0$$

(b) Does $f(t) = 4\ln(1+t) + 13$ satisfy the differential equation $f'(t) = 4(1+t)^{-1}$?

(c) Show that $f(t) = \sqrt{t^2+3}$ is a solution of the IVP $f'(t) = t(f(t))^{-1}$, with $f(1) = 2$.

Continued on next page

6. Describe the following events as initial value problems (i.e., in each case write down a differential equation and an initial condition). Do not solve the equations.
(a) Ice starts forming at time $t = 0$. Let $T(t)$ be the thickness of the ice at time t. The rate at which ice is formed is inversely proportional to the square of its thickness.

(b) At time $t = 0$ somebody starts spreading some rumour on McMaster campus. Assume that there are $15,000$ students on the campus, and denote by $S(t)$ the number of people who have heard the rumour at time t. The rate of increase in the number of people who have heard the rumour is proportional to the number of people who have heard it and to the number of people who haven't heard it yet.

(c) A pie, initially at the temperature of 20^oC, is put into an 300^oC oven. Let $T(t)$ be the temperature of the pie at time t. The temperature of the pie changes proportionally to the difference between the temperature of the oven and the temperature of the pie.

Continued on next page

7. (a) Question 4, page 484 (geese) 4, page 429 (elephants).

(b) Question 6, page 484 (geese) 6, page 429 (elephants).

Continued on next page

8. (a) What is the difference between an autonomous and a pure-time differential equation?

(b) Give an example of a differential equation which is both autonomous differential equation and a pure-time differential equation.

(c) Give an example of a differential equation which neither autonomous differential equation nor a pure-time differential equation.

(d) Read Example 7.1.2 (geese) 6.1.2 (elephants). Sketch the graph of $P(t)$ for $t \geq 0$ (you will need to consider a combination of cases $k > 0$ or $k < 0$ with $M > 0$ or $M < 0$).

Continued on next page

9. Read Example 7.1.4 (geese) 6.1.4 (elephants). Using what is done there, solve the initial value problem $P'(t) = 0.02P(t)$, $P(0) = 240$.

THE END

Math 1LS3	**Assignment 20**

Section 7.1 (geese) 6.1 (elephants) Euler's Method

1. (a) What is Euler's Method used for?

2. In this exercise we get familiar with the notation used in Euler's method. We consider a pure-time differential equation $f'(t) = G(t)$, with initial condition $f(t_0) = y_0$. The step size is denoted by Δt.

If $f'(t) = e^{-2t} + t^3$, what is $G(t)$?

If $f'(t) = \sec(2t - 4)$, what is $G(t)$?

If $f'(t) = \ln(t^2 + 1)$, and $f(2) = -2$, what are $G(t)$, t_0 and y_0?

If $f'(t) = 3\sin(4t)$, and $f(\pi) = 1$, what are $G(t)$, t_0 and y_0?

Given that $f(2) = -2$ and $\Delta t = 0.1$, find the values of t_0, t_1, t_2, t_3, and t_4 for which we compute the approximations.

Given that $f(0) = 5$ and $\Delta t = 0.25$, find the values of t_0, t_1, t_2, t_3, and t_4 for which we compute the approximations.

Continued on next page

3. (a) Given that $f'(t) = \ln(t^2 + 1)$, $f(2) = -2$, and $\Delta t = 0.1$, write down the formulas [look at algorithm 7.1.1 on page 467 (geese) 6.1 on page 413 (elephants)] for Euler's Method; include initial conditions. Round off to three decimal places.

(b) Compute t_1 and y_1.

Compute t_2 and y_2.

Compute t_3 and y_3.

(c) What is the meaning of y_1 in (b)?

What is the meaning of y_2 in (b)?

What is the meaning of y_3 in (b)?

Continued on next page

4. Consider the initial value problem $f'(t) = 3t^2$, $f(0) = 4$.

(a) Compute the first three steps of Euler's Method with step size $\Delta t = 0.5$.

(b) Solve the given initial value problem algebraically.

(c) Plot the function in (b) and the values you obtained in (a) in the same coordinate system.

Continued on next page

5. Consider the initial value problem $P'(t) = 0.4P(t)$, $P(0) = 1000$.

(a) Compute the first four steps of Euler's Method with step size $\Delta t = 1$. Round off to one decimal place.

(b) Show that $P(t) = 1000e^{0.4t}$ is the solution of the given initial value problem.

(c) Plot the function in (b) and the values you obtained in (a) in the same coordinate system.

THE END

Math 1LS3	**Assignment 21**

Section 7.2 (geese) 6.2 (elephants): Differential equations and antiderivatives

1. (a) Define: $F(x)$ is an antiderivative of $f(x)$

(b) Read Example 7.2.2 (geese) 6.2.2 (elephants). Using the same idea, show that

$$\int x^2 e^{2x}\, dx = \frac{1}{4}(1 - 2x + 2x^2)e^{2x} + C$$

(c) Is it true that $\int \arctan x\, dx = \dfrac{1}{1+x^2} + C$?

Continued on next page

2. Read Example 7.2.12 (geese) 6.2.13 (elephants). Apply what you learned to solve

$$\frac{dA}{dt} = 24.6t^3, \quad A(0) = 10.$$

3. Find the function $f(x)$, knowing that $f'(x) = 4/x$, $x < 0$, and $f(-2) = 4$.

Continued on next page

4. Find the function $f(x)$, knowing that $f'(x) = 6^x - 4$, and $f(3) = 12$.

5. Find the following indefinite integrals.

(a) $\int \frac{1}{4}\, dx$

(b) $\int \sqrt{7}\, dx$

(c) $\int (2^x + x^2)\, dx$

Continued on next page

6. Find the following indefinite integrals (by trial and error).

(a) $\int (2\cos(x/3) + 4\cos(3x))\, dx$

(b) $\int (e^x + e^{2x} + e^{x/2})\, dx$

(c) $\int \frac{1}{1+7x}\, dx$

(d) $\int (1+7x)^4\, dx$

THE END

Math 1LS3	**Assignment 22**

Section 7.3 (geese) 6.3 (elephants): Definite integral and area

1. (a) Explain, in three sentences or less, the idea of using approximating rectangles to calculate the area.

(b) What is a Riemann sum?

(c) Rephrase Definition 7.3.1 (geese) 6.2 (elephants), using words instead of symbols.

(d) What is the relation between a definite integral and area?

(e) What is the point of illustrations in Figures 7.3.25 and 7.3.26 (geese) 6.3.25 and 6.3.26 (elephants)?

Continued on next page

2. (a) Read Example 7.3.8 (geese) 6.3.8 (elephants). Using the same idea, find the value of the integral $\int_0^{10} (3x - 2)\, dx$.

(b) Read Example 7.3.10 (geese) 6.3.10 (elephants). Using the same idea, find a and b so that $\int_a^b \sin 2x\, dx = 0$.

Continued on next page

3. Estimate the area under the graph of the function $f(x) = 2x + 3$ on $[1, 4]$ using the following (make a sketch in each case)
(a) three rectangles and right endpoints

(b) three rectangles and left endpoints

(c) three rectangles and midpoints.

Continued on next page

4. Estimate the area under the graph of the function $f(x) = x^2 + 1$ on $[0, 4]$ using the following (make a sketch in each case)
(a) four rectangles and right endpoints

(b) four rectangles and left endpoints

(c) four rectangles and midpoints.

Continued on next page

5. Suppose that you computed the left sum with 5 subintervals for the function $y = e^{-x}$ on $[0, 2]$. Is that sum smaller or larger than the area of the region under $y = e^{-x}$ and over $[0, 2]$? Explain your answer. Sketch the function and the 5 rectangles that form the left sum.

Continued on next page

6. Exercise 34, page 508 (geese) 26, page 451 (elephants).

7. Exercise 40, page 508 (geese) 30, page 451 (elephants).

Continued on next page

8. Read Example 7.3.13 (geese) 6.3.13 (elephants). Then solve Exercise 58, page 509 (geese) 34, page 451 (elephants).

9. Explain the point of Example 7.3.14 (geese) 6.3.14 (elephants).

THE END

Practice with integration: 7.2–7.4 (geese) 6.2–6.4 (elephants)

1. Find the following integrals.

(a) $\displaystyle\int \frac{1}{\sqrt{x}}\,dx$

(b) $\displaystyle\int_1^2 \frac{12}{x^2}\,dx$

(c) $\displaystyle\int \left(3 - \frac{7}{\sqrt[4]{x}}\right)\,dx$

(d) $\displaystyle\int_0^4 (7-2x)^2\,dx$

(e) $\displaystyle\int \frac{x^3-3x+1}{x}\,dx$

(f) $\displaystyle\int \frac{(x^2-3)^2}{7x^3}\,dx$

Continued on next page

(g) $\int \frac{1}{23}\,dx$

(h) $\int \frac{1}{23}\,dt$

(i) $\int \frac{1}{23}\,du$

(j) $\int_1^{11} \ln 4\,dt$

(k) $\int \left(e^x - 2^x + 6^x\right)\,dx$

(l) $\int_0^1 4e^x\,dx$

(m) $\int_0^1 4e^t\,dt$

(n) $\int_0^1 4e^t\,dt$

(o) $\int \left(4e^2 - 24x\right)\,dx$

(p) $\int \frac{14^x - 3}{7}\,dx$

Continued on next page

(q) $\int \cos x \, dx$

(r) $\int_0^1 \cos 3x \, dx$

(s) $\int \cos(-4x) \, dx$

(t) $\int \sec^2 x \, dx$

(u) $\int_0^1 \sec^2 3x \, dx$

(v) $\int \sec^2(-4x) \, dx$

(w) $\int \sec x \tan x \, dx$

(x) $\int_0^1 \sec 4x \tan 4x \, dx$

(y) $\int (\sin 2x - \cos 3x + 4) \, dx$

(z) $\int_0^1 (10 \sin(\pi x) - 2) \, dx$

Continued on next page

(aa) $\int_0^1 \frac{1}{1+x^2}\,dx$

(ab) $\int \left(2 - \frac{4}{1+x^2}\right)\,dx$

(ac) $\int \frac{1}{1+4x^2}\,dx$

(ad) $\int \frac{5}{1+6x^2}\,dx$

(ae) $\int 45(1-x^2)^{-1/2}\,dx$

(af) $\int \left(\frac{3}{\sqrt{1-x^2}} + \sin x\right)\,dx$

(ag) $\int \frac{1}{\sqrt{1-9x^2}}\,dx$

Continued on next page

2. Evaluate each integral by interpreting it as area or as a difference of areas.

(a) $\int_0^{13} \frac{1}{4}\, dx$

(b) $\int_{-13}^{13} \frac{1}{4}\, dx$

(c) $\int_{-13}^{13} \left(-\frac{1}{4}\right) dx$

(d) $\int_0^{3} \sqrt{9-x^2}\, dx$

(e) $\int_{-3}^{3} \sqrt{9-x^2}\, dx$

Continued on next page

(f) $\int_1^5 (3x-1)\,dx$

(g) $\int_0^5 (3x-1)\,dx$

(h) $\int_4^6 |2x-6|\,dx$

(i) $\int_2^6 |2x-6|\,dx$

(j) $\int_0^{4\pi} \sin x\,dx$

(k) $\int_{-0.3}^{0.3} \tan x\,dx$

THE END

Sections 7.4–7.6 (geese) 6.4–6.6 (elephants): Calculating definite integrals, area

1. Evaluate the following definite integrals.

(a) $\displaystyle\int_0^{\pi} (\sin x - 2\cos x)\, dx$

(b) $\displaystyle\int_1^{9} \frac{1}{\sqrt{x}}\, dx$

(c) $\displaystyle\int_0^{1} \frac{6}{1+x^2}\, dx$

Continued on next page

(d) $\int_0^1 11e^x \, dx$

(e) $\int_{-2}^{-1} x^{-1} \, dx$

(f) $\int_1^2 \frac{x^2+1}{x} \, dx$

Continued on next page

2. Find the following integrals.

(a) $\displaystyle\int \frac{1+2x}{(x^2+x+2)^2}\,dx$

(b) $\displaystyle\int \frac{e^x}{3-e^x}dx$

Continued on next page

(c) $\int \frac{(1+\sqrt{x})^2}{\sqrt{x}} dx$

(d) $\int_1^2 \frac{(\ln x)^4}{x}\, dx$

Continued on next page

3. Evaluate the following integrals.

(a) $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx$

(b) $\displaystyle\int_0^1 \frac{x}{1+x^2}\,dx$

(c) $\displaystyle\int_0^1 \frac{x^2}{1+x^2}\,dx$ (Hint: start with long division.)

Continued on next page

4. Evaluate $\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$.

5. Compute the area of the region bounded by the curves $y = 1/x$, $y = 1/x^2$, and $x = 3$.

Continued on next page

6. (a) Sketch the region bounded by the curves $y = e^x$, $y = x + 1$, $x = -2$ and $x = 1$ and set up the formula for its area.

(b) Evaluate the integral in (a).

7. Find the area between the curves $y = 2x$ and $y = x^2$.

Continued on next page

8. (a) Sketch the region bounded by the curves $y = \sin x$, $y = \sin 2x$, $x = \pi/4$ and $x = \pi/2$ and set up the formula for its area.

(b) Evaluate the integral in (a).

THE END

Section 7.5 (geese) 6.5 (elephants): Integration by parts

Find the following integrals.

1. $\int x \ln x \, dx$

2. $\int x \sin x \, dx$

Continued on next page

3. $\int_0^2 xe^{-3x}\,dx$

4. $\int x\cos(\pi x)\,dx$

Continued on next page

5. $\int \arcsin(2x)\, dx$

6. $\int_0^1 \arctan x\, dx$

Continued on next page

7. $\int \ln x \, dx$

8. $\int_1^2 (2x+1)e^x \, dx$

Continued on next page

9. $\int x^2 e^{-x} \, dx$

10. $\int x^2 \cos x \, dx$

THE END

Chapter 7 (geese) 6 (elephants): Concepts

1. (a) Find the solution of the differential equation $f'(t) = kf(t)$, $f(0) = a$ (a is a given constant).

(b) What is an initial value problem? What is a solution of an initial value problem?

(c) How many antiderivatives does a continuous function have? Find all antiderivatives of the function $f(x) = x^{-10}$.

(d) Find all antiderivatives of the function $f(x) = 0$.

(e) State the constant product rule for integrals.

Continued on next page

2. (a) Is it true that $\int f(x)g(x)\,dx = \left(\int f(x)\,dx\right)\left(\int g(x)\,dx\right)$? Explain.

(b) Without calculating the integral, say whether $\int_{-3}^{7} e^{-x}\,dx$ is positive or negative.

(c) What is the point of Example 7.3.6 (geese) 6.3.6 (elephants)?

(d) What is net area? How does it relate to the definite integral?

Continued on next page

3. (a) Restate property (3) of the definite integral (see page 504 (geese) 447 (elephants)) in your own words.

(b) Read Example 7.4.13 (geese) 6.4.13 (elephants). How much does the fish grow in the first 10 years of its life?

(c) Explain how to find the area of the region bounded by two curves.

Continued on next page

4. What does the definite integral $\int_a^b f(t)\,dt$ represent, if

(a) $f(t)$ is the rate of change of the pressure in a blood vessel at time t.

(b) $f(t)$ is the speed of a car at time t.

(c) $f(t)$ is the rate of change of the height of a tree at time t.

(d) $f(t)$ is the acceleration of a falling object at time t.

(e) $f(t)$ is the rate of increase in the number of people infected with a flu at time t.

THE END

Math 1LS3	**Assignment 27**

Sections 7.5–7.7 (geese) 6.5–6.7 (elephants): Integration Using Taylor Polynomials, Applications and Improper Integrals

1. (a) Find the degree 3 Taylor polynomial of $f(x) = \arctan x$ at $x = 0$.

(b) Using (a), find an approximation of $\int_0^1 \arctan x \, dx$.

(c) Compare your answer in (b): in Example 7.5.15 (geese) 6.5.15 (elephants) using integration by parts we show that $\int_0^1 \arctan x \, dx = \frac{\pi}{4} - \frac{\ln 2}{2} \approx 0.439$.

Continued on next page

2. Using a quadratic approximation for e^x at $x = 0$, find an approximation for $\int_{0.1}^{1} \frac{e^x}{x^2}\, dx$.

3. Read Example 7.5.18 (geese) 6.5.18 (elephants).

(a) Find $T_3(x)$ for $f(x) = e^x$ at $x = 0$ and use it to approximate $e^{-x^2/2}$.

(b) Find an approximation of $\int_0^1 e^{-x^2/2}\, dx$ using the polynomial that you obtained in (a).

Continued on next page

4. (a) Find the area between $y = x^3$ and $y = \sqrt{x}$.

(b) Find the area between $y = x^3$ and $y = \sqrt{x}$ for $0 \leq x \leq 2$.

Continued on next page

5. (a) Find the average value of $f(x) = x^2$ on $[0, 1]$.

(b) Find the average value of $f(x) = x^3$ on $[0, 1]$.

(c) Your answer in (a) should be larger than the answer in (b). Explain why.

6. (a) Find the average value of $f(x) = x^2$ on $[1, 2]$.

(b) Find the average value of $f(x) = x^3$ on $[1, 2]$.

(c) Your answer in (a) should be smaller than the answer in (b). Explain why.

Continued on next page

7. Find the following improper integrals.

(a) $\displaystyle\int_0^\infty \frac{1}{(1+2x)^{3/2}}\,dx$

(b) $\displaystyle\int_{10}^\infty \frac{1}{x^2}\,dx$

(c) $\displaystyle\int_1^\infty e^{-0.5x}\,dx$

Continued on next page

8. The density of monkeys in Kruger National Park in South Africa is given by the function $f(x) = 0.003x(248 - x)$ monkeys per kilometre, where x is the distance in km from the main entrance into the park.

(a) Find the average density of monkeys from the main entrance into the park to the location 100 km away from it.

(b) Find the total number of monkeys from the main entrance into the park to the location 100 km away from it.

THE END

Math 1LS3	**Assignment 28**

Practice with improper integrals, section 7.7 (geese) 6.7 (elephants)

1. Give a reason (reasons) why each of the following integrals is improper.

(a) $\displaystyle\int_2^4 \frac{x^3}{x-3}\,dx$

(b) $\displaystyle\int_0^1 x^{-1/2}\,dx$

(c) $\displaystyle\int_0^\infty x^{-1/2}\,dx$

(d) $\displaystyle\int_{-10}^\infty e^{-2x}\,dx$

(e) $\displaystyle\int_2^{-4} \frac{1}{x(x+3)}\,dx$

(f) $\displaystyle\int_{-\infty}^\infty x^{-3}\,dx$

Continued on next page

2. Determine whether or not the improper integrals in (a) and (b) converge.

(a) $\displaystyle\int_0^\infty \frac{1}{2x+4}\,dx$

(b) $\displaystyle\int_{-2}^0 \frac{1}{2x+4}\,dx$

(c) Explain why the integral $\displaystyle\int_{-2}^\infty \frac{1}{2x+4}\,dx$ is divergent.

Continued on next page

3. Determine whether or not the following improper integrals converge.

(a) $\int_0^\infty e^{-10x}\,dx$

(b) $\int_0^\infty xe^{-x^2}\,dx$

Continued on next page

(c) $\int_{-\infty}^{1} 2^x \, dx$

(d) $\int_{-\infty}^{0} \frac{1}{1+x^2} \, dx$

Continued on next page

(e) $\displaystyle\int_0^4 \frac{1}{x-4}\,dx$

(f) $\displaystyle\int_0^4 \frac{1}{(x-4)^2}\,dx$

THE END

Integrals and Volumes

1. Find the volume of the solid obtained by rotating the region R bounded by the given curves about the given axis.

(a) $y = 4 - 2x$, $y = 0$ and $x = 0$; about the x-axis.

(b) $y = 4 - 2x$, $y = 0$ and $x = 0$; about the y-axis.

(c) $y = \sqrt{\sin x}$, $y = 0$, $x = 0$, and $x = \pi/2$; about the x-axis.

Continued on next page

(d) $y = x$ and $y = x^3$ in the first quadrant; about the x-axis

(e) $y = x$ and $y = x^3$ in the first quadrant; about the y-axis

Continued on next page

(f) $y = 8 - x$, $y = 1$, $x = 2$, and $x = 5$; about the x-axis.

(g) $y = x^2$, $y = 18 - x^2$; about the x-axis.

Continued on next page

2. Describe the solid of revolution whose volume is given by each integral.

(a) $\pi \int_0^5 x^2 dx$

(b) $\int_0^5 x^2 dx$

(c) $\pi \int_{1/2}^1 \left(x^{-1} - x^3\right) dx$

THE END

Math 1LS3	Assignment 30

Application (Spread of Ebola Virus)

This assignment is related to the paper Althaus CL. *Estimating the Reproduction Number of Ebola Virus (EBOV) During the 2014 Outbreak in West Africa.* PLOS Currents Outbreaks. 2014 Sep 2. Edition 1. After Introduction section, we read:

Methods

The transmission of EBOV follows SEIR (susceptible-exposed-infectious-recovered) dynamics and can be described by the following set of ordinary differential equations (ODEs):[2]

$$\frac{dS}{dt} = -\beta(t)SI/N,$$
$$\frac{dE}{dt} = \beta(t)SI/N - \sigma E,$$
$$\frac{dI}{dt} = \sigma E - \gamma I,$$
$$\frac{dR}{dt} = (1-f)\gamma I.$$

After transmission of the virus, susceptible individuals S enter the exposed class E before they become infectious individuals I that either recover and survive (R) or die. $1/\sigma$ and $1/\gamma$ are the average durations of incubation and infectiousness. The case fatality rate is given by f. The transmission rate in absence of control interventions is constant, i.e., $\beta(t) = \beta$. After control measures are introduced at time $\tau \leq t$, the transmission rate was assumed to decay exponentially at rate k:[3]

$$\beta(t) = \beta e^{-k(t-\tau)},$$

i.e., the time until the transmission rate is at 50% of its initial level is $t_{1/2} = \ln(2)/k$. Assuming the epidemic

1. Read the above excerpt from the sentence "The transmission rate ..." until the end. Consider the exponential decay formula for $\beta(t)$ and answer the following questions.

(a) Identify all parameters. State whether each parameter is positive or negative.

(b) What is the initial level of the transmission rate?

Continued on next page

(c) Justify the statement about the transmission rate made in the last line:

the time until the transmission rate is at 50% of its initial level is $t_{1/2} = \ln(2)/k$.

(d) When will the transmission rate reach 10% of its initial value?

(e) In the same coordinate system, sketch the graphs of the two versions of $\beta(t)$ mentioned: the constant function, and the exponentially decreasing function. Label β and τ in your diagram.

Continued on next page

(f) What is the limit of $\beta(t)$ as $t \to \infty$? Does it make sense?

(g) Find the total number of transmissions (thus potential ebola infections) during the first 30 days from the introduction of control measures.

Continued on next page

2. Read the whole excerpt on page 1 of this assignment and focus on the given system of differential equations.

(a) List all parameters and state their meaning. (The parameter N, whose meaning is not mentioned in the excerpt, represents the total population.)

(b) Explain why the model predicts that the number of susceptible individuals will decrease. Does it make sense?

(c) What is the meaning of the two terms on the right side of the equation for dE/dt, i.e., how do they affect $E(t)$?

(d) What is the meaning of the two terms on the right side of the equation for dI/dt, i.e., how do they affect $I(t)$?

Continued on next page

(e) What does the last equation (about dR/dt) say about the number of people who recover and survive?

(f) Assume that $I(t) = e^{-\alpha t}$ and $R(0) = 0$. Find the solution of the last differential equation, i.e. find a formula for $R(t)$.

THE END

Sections 3.1, 3.2 (geese) 2.1, 2.2 (elephants): Discrete-time dynamical systems

1. Write the updating function associated with the following discrete dynamical systems.

(a) $P_{t+1} = 4.6 - 2P_t$.

(b) $k_{t+1} = 2k_t^2 - 1.44$.

(c) $m_{t+1} = \dfrac{m_t^2}{m_t + 2}$.

(d) $P_{t+1} = \dfrac{3.2}{P_t^3}$.

2. Find the backwards discrete-time dynamical system for each of the following systems.

(a) $P_{t+1} = 4.6 - 2P_t$.

(b) $k_{t+1} = 2k_t^2 - 1.44$.

(c) $N_{t+1} = \dfrac{N_t}{N_t + 2}$.

(d) $P_{t+1} = \dfrac{3.2}{P_t^3}$.

Continued on next page

3. Find and then graph the first five values of the following discrete-time dynamical systems.

(a) $P_{t+1} = 0.4P_t$, starting from $P_0 = 1000$, and $Q_{t+1} = 0.4Q_t$, starting from $Q_0 = 2000$. Sketch both P_t and Q_t in the same coordinate system.

(b) $m_{t+1} = 0.5m_t + 1$, starting from $m_0 = 20$.

Continued on next page

4. Consider the discrete-time dynamical system $Q_{t+1} = 0.4Q_t$, $Q_0 = 2000$ from the previous question.

(a) Find the formula for Q_t as a function of t.

(b) Using your formula from (a), find $Q(5)$ and compare with your answer to the previous question.

(c) Find $Q(14)$.

5. Exercise 60, page 129 (geese) 62, page 110 (elephants) in the textbook.

Continued on next page

6. Starting with the given initial condition, cobweb the following systems for four steps. Then calculate the values algebraically and compare with your diagram.

(a) $P_{t+1} = 1.2P_t$, starting from $P_0 = 10$.

(b) $m_{t+1} = 0.5m_t + 2$, starting from $m_0 = 40$.

Continued on next page

7. For the given discrete-time dynamical system, find and graph the updating function and then cobweb starting with the given initial condition.

(a) $N_{t+1} = 0.4N_t + 3$, starting from $N_0 = 2$; cobweb for two steps.

(b) $m_{t+1} = -0.5m_t + 2$, starting from $m_0 = 0$; cobweb for three steps.

(c) $m_{t+1} = -m_t^2 + 2$, starting from $m_0 = 1.2$; cobweb for two steps.

Continued on next page

8. Find the equilibria of the following systems.

(a) $x_{t+1} = 3x_t - 1$

(b) $m_{t+1} = -0.5m_t + 2$

(c) $m_{t+1} = \dfrac{0.5m_t}{m_t + 1}$

THE END

Chapter 3 (geese) 2 (elephants): Concepts and applications

1. (a) What is a discrete-time dynamical system? What is an updating function?

(b) What is the solution of a dynamical system?

(c) Read Example 3.1.4 (geese) 2.1.4 (elephants). Starting from $M_0 = 4$, calculate M_1, M_2, M_3, M_4, and M_5.

(d) If $p_{t+1} = 0.57 p_t$ and $p_0 = 12$, what is p_{100}?

(e) If $p_{t+1} = p_t + 0.57$ and $p_0 = 12$, what is p_{100}?

Continued on next page

2. Describe the following events as discrete-time dynamical systems (i.e., state the dynamic rule and identify the initial condition).

(a) The number of deer in a forest increases by 4.5 percent per year. Initially there are 120 deer.

(b) A population of bacteria triples every hour. Every hour, after reproduction, 1,000 bacteria are removed. The population starts with 3,000 bacteria.

(c) A population of bacteria triples every hour. Every hour, before reproduction, 1,000 bacteria are removed. The population starts with 3,000 bacteria.

(d) A tree grows by 4 metres per year. At the start of the experiment, the tree is 1.5 metres high.

(e) A patient's body absorbs 30 % of medication per hour. Every hour, the patient is given 0.5 units of medication. The starting dosage was 2 units.

Continued on next page

3. (a) What are the domain and the range of the updating function in Example 3.1.3 (geese) 2.1.3 (elephants)?

(b) What are the domain and the range of the updating function in Example 3.1.4 (geese) 2.1.4 (elephants)?

(c) The dynamical system system $l_{t+1} = 1.1 l_t + 0.2$, $l_0 = 1.2$ describes the height of a plant in metres, where the time t is measured in years. Convert the dynamical system so that the height is given in inches.

(d) What is the purpose of cobwebbing?

Continued on next page

4. (a) What is an equilibrium of a dynamical system?

(b) Show that $l^* = 3$ is not an equilibrium of the dynamical system $l_{t+1} = 1.1 l_t + 0.2$, $l_0 = 1.2$

(c) Show that $m^* = 4$ is an equilibrium of the dynamical system $m_{t+1} = 0.25 m_t + 3$, $m_0 = 13$.

(d) Give an example of a dynamical system which has no equilibrium.

(e) Give an example of a dynamical system which has exactly two equilibria.

Continued on next page

5. Read Example 3.2.6 (geese) 2.2.6 (elephants). Is there an equilibrium if -0.6 in the updating function is replaced by -0.9? Replaced by 1.3?

6. Read Example 3.2.9 (geese) 2.2.9 (elephants) and summarize it: in three sentences or less, say what it is about and what the conclusion is.

Continued on next page

7. Read Example 3.3.1 (geese) 2.3.1 (elephants).

(a) If we have three double espressos and a Red Bull at 10pm, how much caffeine will be in our body at midnight?

(b) What information was used to calculate the half-life of caffeine? [Read the introductory text above the example]

(c) Assume that at noon there is no caffeine present in our body. Starting at noon, every hour we consume a double espresso. How much caffeine will be in our body at 6pm?

Continued on next page

8. What is the per capita production? If a population has the per capita production of 1.05, is it increasing or decreasing?

9. A population decreased from 1 million to 1 thousand in 1 year. What is its per capita production rate (in units of new members per member per year)?

10. If the number of individuals falls below certain critical number m, the population faces extinction (since it is no longer capable to avoid damaging effects of inbreeding and is unable to cope with changes in the environment).
In 1990, there were about 5,000 southern mountain caribou in British Columbia. In 2009, only about 1,900 remain. Critical number is $m = 500$.
Calculate the time when the caribou population will fall to the level that will threaten their survival. Assume that the per capita production rate is constant.

Continued on next page

11. Read Examples 3.3.6 (geese) 2.3.6 (elephants) and 3.3.7 (geese) 2.3.7 (elephants) and summarize their conclusion in one sentence.

12. Exercise 16, page 151 (geese) 16, page 132 (elephants).

13. Consider the alcohol consumption model $a_{t+1} = a_t - \dfrac{10.1a_t}{4.2 + a_t} + d$, where d is the constant amount that is consumed every hour. For which values of d is there an equilibrium? For which values is there no equilibrium?

THE END

Sections 6.7, 6.8 (geese) 5.5, 5.6 (elephants): Stability

1. For a given discrete-time dynamical system
(i) graph the updating function
(ii) find equilibria graphically and also calculate them algebraically
(iii) check stability of each equilibrium point by cobwebbing
(iv) check stability using the slope criterion.

(a) $m_{t+1} = 0.53m_t + 6.2$

Continued on next page

(b) $x_{t+1} = 0.4x_t$, where $0 \leq x_t \leq 12$

(c) $x_{t+1} = 1.3x_t - 5$, where $0 \leq x_t \leq 20$

Continued on next page

(d) $p_{t+1} = \dfrac{2p_t}{1 + p_t}$

Continued on next page

2. Sketch (and explain, by cobwebbing) a possible graph of an updating function that

(a) has a corner at an equilibrium and it is unstable

(b) has a corner at an equilibrium and it is stable

(c) has a corner at an equilibrium and it is neither stable nor unstable

Continued on next page

3. Find the inverses of the updating functions involved and discuss the stability of the original and the inverse dynamical systems.

(a) $x_{t+1} = 0.4x_t + 1.2$, where $0 \leq x_t \leq 15$

(b) $p_{t+1} = \dfrac{2p_t}{1 + 0.01p_t}$

Continued on next page

4. (a) Consider the discrete-time dynamical system $x_{t+1} = 1 + 0.8(x_t - 1.1)$ Find the equilibrium and determine its stability using cobwebbing. Compare with the algebraic stability condition.

(b) Same question as (a) for the system $x_{t+1} = 1 - 1.2(x_t - 1.1)$.

Continued on next page

(c) Same question as (a) for the system $x_{t+1} = 1 - 0.8(x_t - 1.1)$.

(d) Same question as (a) for the system $x_{t+1} = 1 + 1.2(x_t - 1.1)$.

THE END

Practice with multiple choice and true/false questions, chapters 1,2 (geese) 0,1 (elephants)

Multiple choice questions: circle ONE answer. No justification is needed; on a test, the question is marked all or nothing.

1. Which formula(s) is/are correct for all x and y for which the functions involved are defined?

(I) $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ (II) $\dfrac{1}{x+y} = \dfrac{1}{x} + \dfrac{1}{y}$ (III) $\ln(x+y) = \ln x + \ln y$

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

2. What is the domain of the function $f(x) = \ln(x-1) + \ln(x+1)$?

(A) $x < -1$ (B) $x > -1$ (C) $x > 0$ (D) $x < 0$

(E) $-1 < x < 1$ (F) $0 < x < 1$ (G) $x > 1$ (H) $x < 1$

Continued on next page

3. Which of the following functions is/are defined for all real numbers?

(I) $(x^4 - 1)^{-1}$ (II) $(x^4 - 1)^2$ (III) $(x^4 - 1)^{1/2}$

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

4. The solutions of the equation $\csc x = 2$ in the interval $[0, 2\pi]$ are

(A) no solutions (B) $\pi/4, 5\pi/4$ (C) $\pi/4, 7\pi/4$ (D) $\pi/4, 3\pi/4$
(E) $\pi/6, 5\pi/6$ (F) $\pi/6, 7\pi/6$ (G) $\pi/6, 11\pi/6$ (H) $\pi/3, 5\pi/3$

5. The wingspan W (in centimetres) of certain tropical birds is related to their body mass M (in grams) by $W = 2.37M^{2/3}$. Which of the following statements is/are true?

(I) If M increases, so does W

(II) If M doubles, so does W

(III) W is proportional to $M^{2/3}$

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

Continued on next page

Identify each statement as true or false (circle your choice). On a test, you might, or might not be asked to justify your answer.

6. The function $f(x) = \cos(\pi x)$ has period 2.

TRUE FALSE

7. If T is inversely proportional to S, then S is inversely proportional to T.

TRUE FALSE

8. $x^{0.67} \leq x^{0.88}$ for all positive numbers x.

TRUE FALSE

9. Since $\tan(-\pi/4) = -1$, it follows that $\arctan(-1) = -\pi/4$.

TRUE FALSE

Continued on next page

10. The formula $e^{2x} = 2e^x$ is true (i.e., holds for all real numbers).

TRUE FALSE

11. There is a real number x for which $e^{2x} = 2e^x$ is true.

TRUE FALSE

12. $\arctan(\pi/2)$ is not defined.

TRUE FALSE

13. The following calculation is correct:

$$\arcsin \pi = \frac{1}{\sin \pi} = \frac{1}{-1} = -1.$$

TRUE FALSE

THE END

Practice with multiple choice and true/false questions, chapters 4,5 (geese) 3,4 (elephants)

Multiple choice questions: circle ONE answer. No justification is needed; on a test, the question is marked all or nothing.

1. Which of the following functions is/are not continuous at $x = 0$?

(I) $f(x) = \ln(e^x + 1)$ (II) $f(x) = \ln(e^x - 1)$ (III) $f(x) = \ln(e^x)$

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

2. If $f(x) = \sin(\cos(\sin(\cos x)))$ then $f'(0) =$

(A) not defined (B) ∞ (C) -1 (D) 1

(E) π (F) $\pi/2$ (G) $\pi/4$ (H) 0

Continued on next page

Identify each statement as true or false (circle your choice). On a test, you might, or might not be asked to justify your answer.

3. The fact that $\lim_{x\to 2} f(x) = 3$ implies that $f(x)$ is continuous at 2.

TRUE FALSE

4. The fact that $\lim_{x\to 2} f(x) = 3$ implies that $f(2) = 3$.

TRUE FALSE

5. If $\lim_{x\to\infty} f(x) = 4$ then there is no real number x where $f(x) = 4$.

TRUE FALSE

6. $f(x) = \arctan\sqrt{x}$ is a continuous function for all $x \geq 0$.

TRUE FALSE

Continued on next page

7. The function $f(x) = x^{-2}$ is the leading behaviour at infinity of the function $x^{-3} + x^{-2}$.

TRUE FALSE

8. If $f(x) > g(x)$ for all $x > 1$ then $\lim_{x\to\infty} f(x) > \lim_{x\to\infty} g(x)$

TRUE FALSE

9. If $n(t)$ gives the number of monkeys infected with monkey flu as a function of time in days, then the relative rate of change $n'(t)/n(t)$ is given in number of monkeys/day.

TRUE FALSE

10. The largest slope of the graph of $y = \cos x$ is 1.

TRUE FALSE

THE END

Practice with multiple choice and true/false questions, chapter 6 (geese) 5 (elephants)

Multiple choice questions: circle ONE answer. No justification is needed; on a test, the question is marked all or nothing.

1. It is known that $f'(a) = 0$ and $f''(a) > 0$. Which statements is/are true?

(I) $f(x)$ is concave down near a

(II) a is an inflection point of the graph of $f(x)$

(III) $f(x)$ has a relative extreme value at a

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

2. It is known that a is in the domain of $f(x)$ and $f'(a) = 0$. Which statements is/are true?

(I) a is a critical point of $f(x)$

(II) $f(x)$ has a horizontal tangent at a

(III) $f(x)$ has a relative extreme value at a

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

Continued on next page

Identify each statement as true or false (circle your choice). On a test, you might, or might not be asked to justify your answer.

3. If $f'(5) = 0$ then $f(x)$ has a local extreme value at $x = 5$.

TRUE FALSE

4. If $f(x)$ is differentiable at $x = 5$ and has a local extreme value at $x = 5$, then $f'(5) = 0$.

TRUE FALSE

5. The function $f(x) = x^2 \sin x + e^{x^2-1}$ has no absolute maximum on the interval $[-2, 7]$.

TRUE FALSE

Continued on next page

6. If $f(x) = \dfrac{e^{2x} + x - x^4}{x^3 + 5}$ then $f(x)_\infty = \dfrac{e^{2x}}{x^3}$.

TRUE FALSE

7. The number $m^* = -2$ is a stable equilibrium of the system $m_{t+1} = 2m_t^2 + 6m_t + 2$.

TRUE FALSE

8. The function $f(x) = x^4 - 3x^3 - 17x^2 - 11x + 3$ has four critical points.

TRUE FALSE

THE END

Practice with multiple choice and true/false questions, chapter 7 (geese) 6 (elephants)

Multiple choice questions: circle ONE answer. No justification is needed; on a test, the question is marked all or nothing.

1. It is known that $\int_a^b f(x)dx = 4$ and $\int_a^b g(x)dx = 7$. Which of the following statements is/are true?

(I) $\int_a^b (f(x) + g(x))\, dx = 11$

(II) $\int_a^b f(x)g(x)dx = \left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right) = 28$

(III) $\int_a^b (2f(x) - g(x))\, dx = 1$

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

Identify each statement as true or false (circle your choice). On a test, you might, or might not be asked to justify your answer.

2. If $\int q(t)dt = p(t)$, then $q'(t) = p(t)$.

TRUE FALSE

Continued on next page

3. $\int \ln x dx = \frac{1}{x} + C.$

TRUE FALSE

4. If $\int_a^b f(x)dx = 0$, then $f(x) = 0$ on $[a, b]$.

TRUE FALSE

5. If $f(1) = 4$, $f(2) = 5$, and f' is continuous on $[1, 2]$, then $\int_1^2 f'(x)dx = 5 - 4 = 1.$

TRUE FALSE

Continued on next page

6. $\int_a^b f(t)dt = \int_a^b f(u)du = \int_a^b f(x)dx.$

TRUE FALSE

7. The following calculation is correct:

$$\int_{-1}^{1} x^4 dx = x^4\Big|_{-1}^{1} = (1)^4 - (-1)^4 = 0.$$

TRUE FALSE

8. The fact that $\left(\frac{x}{4\sqrt{x^2+4}}\right)' = \frac{1}{(x^2+4)^{3/2}}$ proves that

$$\int \frac{1}{(x^2+4)^{3/2}} dx = \frac{x}{4\sqrt{x^2+4}} + C.$$

TRUE FALSE

Continued on next page

9. If $f(x) = x^{-2}$, $x > 0$, then Riemann sums satisfy $L_7 > M_7 > R_7$.

TRUE FALSE

10. The following calculation is correct:

$$\int x^2 e^{2x} dx = \left(\int x^2 dx\right)\left(\int e^{2x} dx\right) = \frac{x^3}{3}\frac{e^{2x}}{2} + C.$$

TRUE FALSE

11. The definite integral $\int_{1/2}^{1} \ln x dx$ is a positive number.

TRUE FALSE

THE END

Practice with multiple choice and true/false questions, chapter 3 (geese) 2 (elephants)

Multiple choice questions: circle ONE answer. No justification is needed; on a test, the question is marked all or nothing.

1. Which of the three points is/are the equilibrium points of the discrete-time dynamical system $m_{t+1} = 4m_t^2 - 3m_t - 8$?

(I) $m^* = 0$ (II) $m^* = 1$ (III) $m^* = 2$

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

2. The function $f(x) = 3x - 4$ is the updating function of which dynamical system(s)?

(I) $m_{t+1} = \dfrac{1}{3m_t - 4}$ (II) $m_{t+1} = 3(m_t - 4)$ (III) $m_{t+1} = 3m_t$

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

Continued on next page

Identify each statement as true or false (circle your choice). On a test, you might, or might not be asked to justify your answer.

3. The dynamical systems $b_{t+1} = b_t^2$ and $b_{t+1} = b_t^3$ have the same equilibrium points.

TRUE FALSE

4. Cobwebbing the system $m_{t+1} = 1.05m_t$ starting at $m_0 = 1$ will approach the equilibroum value $m^* = 0$.

TRUE FALSE

5. $m^* = 0$ is an equilibrium point of the system $m_{t+1} = f(m_t)$ no matter what the updating function f is.

TRUE FALSE

6. The dynamical system $m_t = \dfrac{1 + m_{t+1}}{1 - m_{t+1}}$ is the backward dynamical system of the system $m_{t+1} = \dfrac{m_t - 1}{m_t + 1}$

TRUE FALSE

THE END

Math functions in context; Chapters 1,2 (geese) 0,1 (elephants)

1. The resistance R of the flow of blood through a blood vessel (assumed to have the shape of a cylindrical tube) is given by

$$R = \frac{Kl(\gamma + 1)^2}{d^4}$$

where l is the length of the vessel, d is its diameter and $\gamma \geq 0$ is the curvature. The positive constant K represents the viscosity of the blood (viscosity is a measure of the resistance of fluid to stress; water has low viscosity, honey has high viscosity).

(a) Using the words "proportional" and "inversely proportional" describe how R depends on the viscosity of the blood, on the length of the vessel and on the diameter of the vessel.

(b) In a graph, show how the resistance R depends on the diameter of a blood vessel. Label the axes.

(c) Does the graph in (b) make sense? Explain why or why not.

Continued on next page

(d) Sketch the graph showing how the resistance R depends on the curvature of the vessel. A straight line (straight vessel) has curvature $\gamma = 0$.

(e) Does the graph in (d) make sense? Explain why or why not.

(f) Describe in words the graph of R, viewed as a function of the length of a blood vessel.

(g) Describe in words the graph of R, viewed as a function of the viscosity of the blood.

Continued on next page

2. The following excerpt is taken from *A novel computational method identifies intra- and inter-species recombination events in Staphylococcus aureus and Streptococcus pneumoniae.* Lisa Sanguinetti, et al. PLoS Computational Biology. 8.9 (Sept. 2012).

> Although any function with a compact support could be used as a weighting function, in the following we have used a modified *rect* function:
>
> $$rect_l = \begin{cases} 1/l & 0 \le x \le l \\ 0 & otherwise \end{cases}$$
>
> where the $1/l$ factor guarantees the normalization

Sometimes (in applied math literature) instead of the traditional notation $f(x)$ for a function f of a variable x, we find the notation f. As well, the symbol for a function is chosen according to context; in this case, it is not f but *rect*. From the given formula (the $0 \le x \le l$ part) we deduce that *rect* is a function of x; thus, we think of it as $rect(x)$. The function depends on the value of the parameter l, which is in the subscript. Hence the notation $rect_l$, or, including the independent variable, $rect_l(x)$.

(a) Assume that the parameter l is positive. Describe in words what the graph of $rect_l$ (keep in mind it is $rect_l(x)$) looks like.

(b) Sketch the graphs of $rect_3$, $rect_{10.3}$ and $rect_{16}$.

(c) Describe what happens to the graph of $rect_l$ as l increases.

Continued on next page

3. The following excerpt is taken from *Multiscale modeling of red blood cell mechanics and blood flow in malaria.* Bruce Caswell et al. PLoS Computational Biology. 7.12 (Dec. 2011).

Membrane macroscopic properties

Extension of the linear analysis of [19] for a regular hexagonal network allows us to uniquely relate the model parameters and the network macroscopic elastic properties (shear, area-compression, and Young's moduli), see [13,17] for details. The derived shear modulus of the membrane is given by

$$\mu_0 = \frac{\sqrt{3}k_B T}{4pl_m x_0}\left(\frac{x_0}{2(1-x_0)^3} - \frac{1}{4(1-x_0)^2} + \frac{1}{4}\right) + \frac{\sqrt{3}k_p(n+1)}{4l_0^{n+1}}, \quad (7)$$

where l_0 is the equilibrium spring length and $x_0 = l_0/l_m$. The area-compression K and Young's Y moduli are equal to $2\mu_0 + k_a + k_d$ and $4K\mu_0/(K+\mu_0)$, respectively.

Reading further, you learn that all quantities involved are positive.

(a) Assume that T is the independent variable (so that all other symbols on the right side of (7) represent parameters). What is the graph of μ_0 as a function of T?

(b) Assume that $x_0 = 1/2$ and $k_p = 0$. Sketch the graph of μ_0 as a function of T.

Continued on next page

(c) Assume that $x_0 = 1/2$ and $k_p = 0$. Sketch the graph of μ_0 as a function of p.

(d) Assume that $x_0 = 1/2$ and $k_p = 0$. Sketch the graph of μ_0 as a function of k_B.

Continued on next page

4. The following excerpt is taken from *The emergence of environmental homeostasis in complex ecosystems.* James G. Dyke and Iain S. Weaver. PLoS Computational Biology. 9.5 (May 2013).

> This analysis assumes that values of K are very large. Numerical simulations allow us to explore the behaviour of the model as K is increased from 1. We find that beyond a threshold value of K, the expected number of times F changes sign and so the expected number of stable points remains constant. The threshold value of K is approximately
>
> $$K \approx \left(\frac{5R}{\sigma_E}\right)^N. \qquad (7)$$

Reading further, you found out that R and σ_E are positive.

(a) Sketch the graph of K as a function of N.

(b) Let $\sigma_E = 10$ and $N = 3$. Sketch the graph of K as a function of R.

Continued on next page

(c) Sketch the graph showing how K depends on σ_E if you know that $N = 1$ and $R = 2$.

(d) Sketch the graph showing how K depends on σ_E if you know that $N = -1$ and $R = 2$.

(e) Assume that $N = 2$. How does K change if R triples?

Continued on next page

(f) Assume that $N = -2$. How does K change if R triples?

(g) Assume that $N = 3$. How does K change if σ_E doubles?

(h) Assume that $N = -1$. How does K change if σ_E doubles?

THE END

Math functions in context; Chapter 2 (geese) 1 (elephants)

1. The interstitial fluid pressure (IFP) p the at a location r mm from the centre of a tumour is given by

$$p = 0.267p_i + \frac{1.3}{r}\frac{\sinh(0.4r)}{\sinh\alpha}$$

where p_i is the atmospheric pressure (assumed constant) and α is a positive parameter which involves the values related to a fluid movement within the tumour.

(a) Searching Wikipedia, you found that $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$. Find the numeric values of $\sinh 0.2$ and $\sinh 3$. Round off the fifth decimal place.

(b) Find p if $p_i = 10$, $r = 0.5$ and $\alpha = 3$. Round off the fifth decimal place.

(c) What is the graph of p as a function of p_i?

Continued on next page

2. The following excerpt is taken from *Delay selection by spike-timing-dependent plasticity in recurrent networks of spiking neurons receiving oscillatory inputs.* Robert R. Kerr, et al. PLoS Computational Biology. 9.2 (February 2013).

> The spike trains in each input group were generated from the group's input intensity function. These are defined for each group of oscillatory inputs as
>
> $$\begin{aligned}\hat{\lambda}_1(t) &= \hat{v}_0 + a \cdot \cos[2\pi f_{\mathrm{m}}(t + \hat{d})], \\ \hat{\lambda}_2(t) &= \hat{v}_0 + a \cdot \cos[2\pi f_{\mathrm{m}}(t + \hat{d} + \hat{d}_{\mathrm{lag}})],\end{aligned} \tag{33}$$
>
> where $\hat{v}_0$ is the mean input rate (in spikes/s), a is the amplitude in the oscillations (in spikes/s), f_{m} is the modulation frequency of the oscillations (in Hz), $\hat{d}$ is the delay of inputs in the first group (in seconds), and $\hat{d}_{\mathrm{lag}}$ is the time lag between the oscillations of the two input groups (in seconds).

(a) List all parameters which appear in the two formulas (33).

(b) What is the period of $\hat{\lambda}_1(t)$?

(c) What is the period of $\hat{\lambda}_2(t)$?

Continued on next page

(d) Assume that $\hat{\mathbf{v}}_0 = 0$, $a = 2.6$, $f_m = 4$ and $\hat{\mathbf{d}} = \pi$. Sketch the graph of $\hat{\lambda}_1(t)$.

(e) Assume that the values of all parameters are positive. Sketch the graph of $\hat{\lambda}_1(t)$.

(f) Assume that the time lag between the oscillations of the two input groups is negative. Explain how to obtain the graph of $\hat{\lambda}_2(t)$ from the graph of $\hat{\lambda}_1(t)$.

Continued on next page

3. The following excerpt is taken from *Tuning the mammalian circadian clock: robust synergy of two loops.* Hanspeter Herzel et al. PLoS Computational Biology. 7.12 (Dec. 2011).

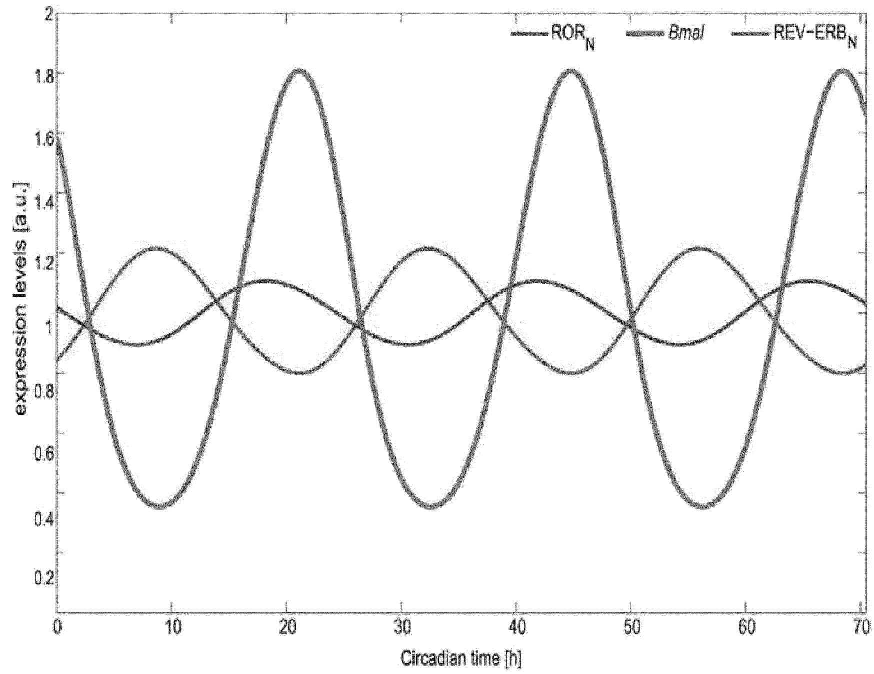


Figure 4. *Bmal* is regulated by the antagonistic action of REV-ERB and ROR. Represented are *in silico* expression profiles for the nuclear protein $REV\text{-}ERB_N$ and ROR_N, and for *Bmal* RNA. The nuclear proteins ROR_N (red) and $REV\text{-}ERB_N$ (blue) recognize and compete for the *cis*-regulatory elements in the *Bmal* (green) promoter region to act, respectively, as positive and negative drivers of *Bmal* expression.
doi:10.1371/journal.pcbi.1002309.g004

Bmal is the curve with the highest amplitude. ROR_N is the curve with the smallest amplitude). $REV\text{-}ERB_N$ is the curve which is initially increasing. Use t to denote the circadian time (in hours).

(a) Estimate the period of the Bmal curve.

(b) Estimate the amplitude and the average of the Bmal curve.

Continued on next page

(c) Find an equation for the Bmal curve using the cosine function.

(d) Find an equation for the REV-ERB$_N$ curve.

(e) It is known that the ROR$_N$ curve has the same period and the same average as the REV-ERB$_N$ curve. Estimate the quantities you need to find an equation for the ROR$_N$ curve.

Continued on next page

4. The following excerpt is taken from *The emergence of environmental homeostasis in complex ecosystems.* James G. Dyke and Iain S. Weaver. PLoS Computational Biology. 9.5 (May 2013).

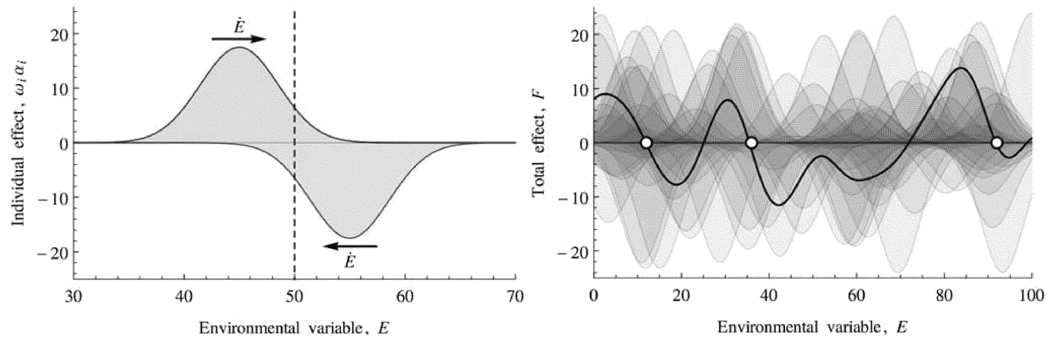


Figure 4. Left: A rein control state is shown. A biotic component that increases the environmental variable, E, counteracts the effects of a biotic component that decreases E. This results in E being regulated around values near the vertical dashed line. The probability of such a rein control pair being present in a population of two biotic components is 1/4. Right: As the number of biotic components is increased up to 100 in this example, a total effect F (solid black line) emerges as it is the sum of the individual biotic effects. Homeostatic stable points (denoted with circles) emerge whenever F undergoes a zero-crossing from left to right. These correspond to rein control homeostatic states.
doi:10.1371/journal.pcbi.1003050.g004

(a) Copy the graph of the function $f(x) = \frac{1}{1+x^2}$ from page 35 in your textbook. Use it to find the formula for the top curve (call it T) in the Individual Effect diagram.

(b) Based on (a), find a formula for the bottom curve (call it B) in the same diagram.

Continued on next page

5. The following excerpt is taken from *Evaluating the adequacy of gravity models as a description of human mobility for epidemic modelling.* James Truscott and Neil M. Ferguson. PLoS Computational Biology. 8.10 (Oct. 2012)

> By setting parameters to zero, we can examine a range of submodels. We look at models of distance interaction of two forms, smooth kernel (SK)
>
> $$f(d)=\begin{cases} \pi & d=0, \\ \dfrac{1}{(1+d/\alpha)^{\gamma}} & d>0. \end{cases}$$

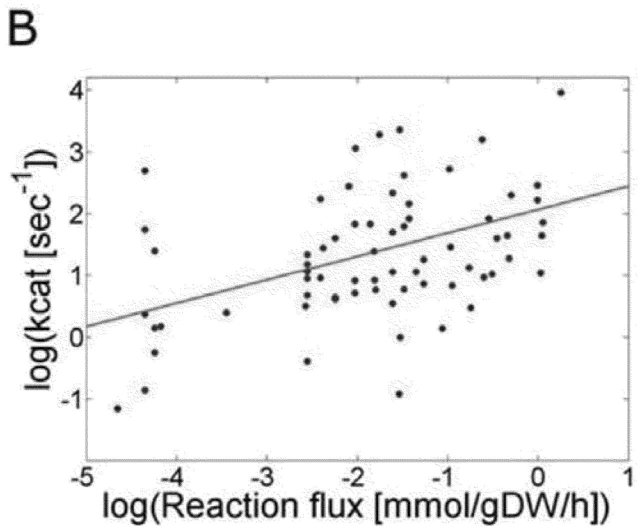
The exponent in the denominator in the second line is γ.

(a) Find $f(0)$ and $f(5)$.

(b) Sketch the graph of the smooth kernel $f(d)$ if $\alpha = 4$ and $\gamma = 2$.

Continued on next page

6. In *Prediction of microbial growth rate versus biomass yield by a metabolic network with kinetic parameters.* Roi Adadi et al. PLoS Computational Biology. 8.7 (July 2012) we find the following diagram



Assume that $\log = \log_{10}$.

(a) Estimate the slope of the line and write its equation in the log-log coordinate system (note that the value at -5 is not zero; assume that it is 0.1; as well, you can assume that the value at 1 is 2.4). Use the symbol k for the function kcat and r for the reaction flux (independent variable).

(b) Eliminate logarithms to obtain an explicit formula for k as a function of r.

THE END

Math functions in context; Chapters 2–5 (geese) 1–4 (elephants)

1. The following excerpt is taken from *Connection between oligomeric state and gating characteristics of mechanosensitive ion channels.* Christoph A. Haselwandter and Rob Phillips. PLoS Computational Biology. 9.5 (May 2013)

The central quantity in this model is the channel opening probability

$$P_o = \frac{1}{1 + e^{\beta(\Delta\mathcal{G} - \tau\Delta A)}}, \qquad (1)$$

where $\beta = 1/k_B T$, in which k_B is Boltzmann's constant and T is the temperature, $\Delta\mathcal{G}$ is the total free energy difference between the open and closed states of MscL, τ is the membrane tension, and ΔA is the area difference between the open and closed channel states.

From the context we figured out that the membrane tension τ and the parameter β are positive numbers.

(a) Consider P_0 as a function of the membrane tension; what is its domain?

(b) Probability is a number between 0 and 1 (representing the range from 0% to 100%). Show that P_0 is indeed a probability.

(c) Assume that $\Delta A > 0$. What is the limit of P_0 as the membrane tension keeps increasing (i.e., as the membrane tension approaches infinity)?

Continued on next page

2. The following excerpt is taken from *Evaluating the adequacy of gravity models as a description of human mobility for epidemic modelling.* James Truscott and Neil M. Ferguson. PLoS Computational Biology. 8.10 (Oct. 2012)

> By setting parameters to zero, we can examine a range of sub-models. We look at models of distance interaction of two forms, smooth kernel (SK)
>
> $$f(d) = \begin{cases} \pi & d = 0, \\ \dfrac{1}{(1 + d/\alpha)^{\gamma}} & d > 0. \end{cases}$$

The exponent in the denominator in the second line is γ.

(a) Assume that $\alpha > 0$ and $\gamma > 0$. What is the limit of the kernel $f(d)$ as $d \to \infty$?

(b) Assume that $\alpha > 0$ and $\gamma < 0$. What is the limit of $f(d)$ as $d \to \infty$?

(c) What is the limit of the kernel $f(d)$ as $d \to 0^+$?

(d) Assume that $f(d) = \pi$ for $d \leq 0$. Is the kernel $f(d)$ continuous at $d = 0$?

Continued on next page

3. The following excerpt is taken from *The laminar cortex model: a new continuum cortex model incorporating laminar architecture.* J. Du, V. Vegh, and D.C. Reutens. PLoS Computational Biology. 8.10 (Oct. 2012).

> the average of membrane potentials of neurons in the element, that is
>
> $$V = \frac{N_e V_e + N_i V_i}{N_e + N_i}$$
>
> where N_e, N_i are the numbers of excitatory and inhibitory neurons and V_e and V_i are the (average) membrane potentials of excitatory and inhibitory neuron populations respectively.

(a) View V as a function of V_i (in which case all remaining symbols on the right side represent parameters). Describe in words the graph of V.

(b) View V as a function of N_i (in which case all remaining symbols on the right side represent parameters). What is the limit of V as N_i increases beyond any bounds (i.e., approaches ∞)?

(c) View V as a function of V_e. What is the limit of V as V_e approaches $-\infty$?

Continued on next page

4. Some models involving Allee effect incorporate the fact that bacteria produce poorly when the population size is small. One such model states that

$$b_{t+1} = \frac{2.7a\, b_t^{\alpha+1}}{1.3 + b_t^2}$$

where the parameters a and α is positive. The population b_t is measured in millions. So $b_t = 2$ means 2 million bacteria.

(a) Identify the per capita production function.

(b) Write down the equation you would need to solve to find the equilibria of the given system. Simplify as much as possible.

(c) Solve the equation in (b) for the values $a = 1$ and $\alpha = 2$. Make sure to eliminate those solutions which do not make sense.

Continued on next page

5. In studying L-form bacteria subpopulation in human gut researchers use the model

$$b_{t+1} = b_t \left(ae^{-mb_t} - de^{-nb_t} + 1 \right)$$

where $a, d, m, n > 0$ and $n - m > 0$.

(a) Identify the per capita production function.

(b) For which values of the parameters a and d does the equilibrium make (biological) sense?

Continued on next page

6. The following excerpt is taken from *Predictive dynamics of human pain perception.* Guillermo A. Cecchi et al. PLoS Computational Biology. 8.10 (Oct. 2012).

> Overfitting of the model was investigated using the Akaike Information Criterion (AIC), which penalizes the measure of goodness of fit with a term proportional to the number of free parameters [31]. When the residual squared error sum (*SS*) is known, the criterion can be written as
>
> $$AIC = n\log\left(\frac{SS}{n}\right) + 2k + C$$
>
> where n is the number of samples, and k the number of parameters. C is a constant

Recall the convention $\log = \log_{10}$. Assume that $SS > 0$.

(a) Find the rate of change of AIC with respect to n.

(b) Find the limit of AIC as the number of samples n approaches ∞.

Continued on next page

7. The generalized linear-quadratic (LQ) model for the percent S of cancer cells surviving radiation treatment states that

$$S(d) = \exp\left(-\alpha n d - \beta n d^2 + \gamma T\right)$$

where $d > 0$ is the dose (in Gray) per treatment of radiation, n is the number of treatments applied (also called fractions), γ is the tumour cell repopulation rate (since new tumour cells grow during treatment) and $T > 0$ is the total treatment duration. The parameters α, β and γ are positive. Recall that $\exp(x) = e^x$.

(a) Assume that $\gamma = 0$. What is the range of S in that case?

(b) Explain why $S(d)$ is a continuous function for $d > 0$.

(c) Let $f(d) = -\alpha n d - \beta n d^2 + \gamma T$ (i.e., f is the function in the exponent). Assume that $\gamma = 0$. Make a sketch of $f(d)$.

Continued on next page

(d) Make a sketch of $f(d)$ in a general case when when $\gamma > 0$.

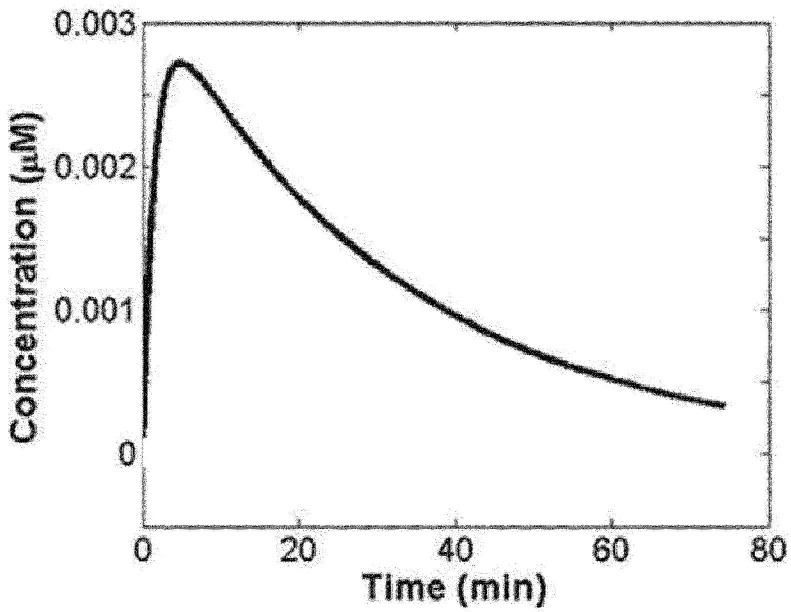
(e) Find all critical points of $S(d)$.

THE END

Math functions in context; Chapters 5,6 (geese) 4,5 (elephants)

1. The following excerpt is taken from *Modeling of pharmacokinetics of cocaine in human reveals the feasibility for development of enzyme therapies for drugs of abuse.* Chang-Guo Zhan and Fang Zheng. PLoS Computational Biology. 8.7 (July 2012).

> A pharmacokinetic model has been developed to examine the effects of a cocaine-metabolizing enzyme in plasma on the time



(a) Consider the function $f(t) = Ate^{-\beta t}$ whose graph is shown in Figure 3.3.41d in your textbook. Find the relative maximum of $f(t)$ and the value of t where it occurs.

Continued on next page

(b) In the given concentration vs time graph estimate the maximum concentration and the time when it occurs. Use these estimates to find A and β, thus obtaining a formula (model) for the concentration as a function of time.

(c) By imitating (a), and (b), obtain another model (i.e., find A and k) for the concentration, based on the function $c(t) = \dfrac{At}{k+t^2}$ with $A, k > 0$ (Figure 3.3.41.e).

Continued on next page

2. The following excerpt is taken from *Hybrid equation/agent-based model of ischemia-induced hyperemia and pressure ulcer formation predicts greater propensity to ulcerate in subjects with spinal cord injury.* Alexey Solovyev et al. PLoS Computational Biology. 9.5 (May 2013).

> Based on this expression of $V(t)$, an explicit solution for $I_2(t)$ can be derived with initial conditions $I_1(0) = I_2(0) = 0$. This solution has the following form
>
> $$I_2(t) = I_{2,rest}(1 + a\exp(-p_1 t) + b\exp(-p_2 t)).$$
>
> Here $I_{2,rest}$, a, b, p_1, and p_2 are constants expressed in terms of R_1, R_2, R_3, R_4, C_1, C_2, V_0.

To avoid exponents (which are written in smaller font, and thus could be hard to read), we sometimes use $exp(x)$ for the exponential function e^x.

Assume that $I_{2,\text{rest}} = 1$, $p_1 = 2$ and $p_2 = 2.4$. Find all critical points of $I_2(t)$. (Your answer will contain a and b).

Continued on next page

3. The following excerpt is taken from *Host resistance, population structure and the long-term persistence of bubonic plague: contributions of a modelling approach in the Malagasy focus.* Fanny Gascuel et al. PLoS Computational Biology. 9.5 (May 2013).

> Using the Next Generation Method, we found
>
> $$R_0 = \sqrt{\frac{\beta K_f}{1 + \frac{d_f}{1 - e^{(-aK)}}}} \qquad (3)$$
>
> The propagation of plague is favoured by a high transmission from fleas to rats (β), which increases the number of infectious rats, and by a high flea carrying capacity of rats (K_f), which increases the number of free fleas, the vectors of the disease (equation (3)). It is also favoured by a high carrying capacity of the rat population, K, and by a high search efficiency of fleas, a, through their direct effect on the probability $1 - e^{-aK}$ that a flea finds a host.

By saying "It (meaning the propagation of plague R_0) is also favoured by a high carrying capacity of the rat population, K, ..." the authors meant to say that the propagation of plague R_0 is an increasing function of the rat population K. Prove that this statement is correct. (All parameters are assumed to be positive.)

Continued on next page

4. The generalized linear-quadratic (LQ) model for the percent S of cancer cells surviving radiation treatment states that

$$S = \exp\left(-\alpha n d - \beta n d^2 + \gamma T\right)$$

where $d > 0$ is the dose (in Gray) per treatment of radiation, n is the number of treatments applied (also called fractions), γ is the tumour cell repopulation rate (i.e., new tumour cells growing during treatment) and $T > 0$ is the total treatment duration. The parameters α, β and γ are positive. Recall that $\exp(x) = e^x$.

(a) Assume that $\gamma = 0$ and view S as a function of the dose d. Show that S is a decreasing function. Does it make sense?

(b) Assume that $\beta = 0$ and view S as a function of the dose d. Show that S a decreasing function.

Continued on next page

(c) Assume that $\alpha = 0$ and view S as a function of the dose d. Show that S a decreasing function.

(d) View S as a function of the number of treatments applied n. Show that S is a decreasing function. Interpret your answer.

Continued on next page

5. The interstitial fluid pressure (IFP) p the at a location r mm from the centre of a tumour is given by

$$p = 0.267p_i + \frac{1.3}{r}\frac{\sinh(0.4r)}{\sinh\alpha}$$

where p_i is the atmospheric pressure (assumed constant) and α is a positive parameter which involves the values related to a fluid movement within the tumour.

Searching Wikipedia, you found that $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$.
The researchers claim that p is an increasing function of r when $r > 2.5$. Justify their claim.

Continued on next page

6. The following excerpt is taken from *Evaluating the adequacy of gravity models as a description of human mobility for epidemic modelling.* James Truscott and Neil M. Ferguson. PLoS Computational Biology. 8.10 (Oct. 2012)

> By setting parameters to zero, we can examine a range of sub-models. We look at models of distance interaction of two forms, smooth kernel (SK)
>
> $$f(d)=\begin{cases} \pi & d=0, \\ \dfrac{1}{(1+d/\alpha)^{\gamma}} & d>0. \end{cases}$$

The exponent in the denominator in the second line is γ.

(a) Find and simplify the formula for the rate of change of the smooth kernel $f(d)$ with respect to d in the case when $d > 0$.

(b) Find and simplify $f''(d)$.

THE END

Math functions in context – Differentiation

1. The following excerpt is taken from *Connection between oligomeric state and gating characteristics of mechanosensitive ion channels.* Christoph A. Haselwandter and Rob Phillips. PLoS Computational Biology. 9.5 (May 2013)

The central quantity in this model is the channel opening probability

$$P_o = \frac{1}{1 + e^{\beta(\Delta\mathcal{G} - \tau\Delta A)}}, \qquad (1)$$

where $\beta = 1/k_B T$, in which k_B is Boltzmann's constant and T is the temperature, $\Delta\mathcal{G}$ is the total free energy difference between the open and closed states of MscL, τ is the membrane tension, and ΔA is the area difference between the open and closed channel states.

From the context we figured out that all parameters are positive numbers.

Show that P_0 is an increasing function of τ.

Continued on next page

2. The following excerpt is taken from *The laminar cortex model: a new continuum cortex model incorporating laminar architecture.* J. Du, V. Vegh, and D.C. Reutens. PLoS Computational Biology. 8.10 (Oct. 2012).

the average of membrane potentials of neurons in the element, that is

$$V = \frac{N_e V_e + N_i V_i}{N_e + N_i}$$

where N_e, N_i are the numbers of excitatory and inhibitory neurons and V_e and V_i are the (average) membrane potentials of excitatory and inhibitory neuron populations respectively.

You know that the numbers N_e and N_i are positive, and the membrane potentials V_e and V_i are negative.

(a) Assume that V is a function of V_e. Find its derivative and interpret your answer.

Continued on next page

For (b) and (c) assume that V is a function of N_e.

(b) Find the derivative of V and interpret your answer.

(c) Find the relation between V_e and V_i which makes the graph of V concave up.

Continued on next page

3. In the paper *Evaluating the adequacy of gravity models as a description of human mobility for epidemic modelling.* James Truscott and Neil M. Ferguson. PLoS Computational Biology. 8.10 (Oct. 2012), we read the following:

> Comparison of the behaviour of the epidemic model on different networks is based on the times to first infection for network nodes from a given initial infection site. Under certain simplifying assumptions, an approximation for the mean time to infection between two nodes can be calculated for the above epidemiological model.
>
> $$\bar{t} \approx \frac{-1}{r}\left\{\ln\left(\frac{\beta_w\beta_h\Lambda_1\varsigma_1}{\beta_w\varsigma_1+\sigma+r}+\frac{\beta_w\beta_h\Lambda_2\varsigma_2}{\beta_h\varsigma_2+\sigma+r}\right)-\gamma\right\} \qquad (5)$$
>
> The parameters $\Lambda_{1,2}$ and $\varsigma_{1,2}$ represent network properties, such as the fraction of journeys between the two nodes. $r=\beta_h+\beta_w+\sigma$ is the epidemic growth rate in a large node. The equation illustrates how

Assume that $\Lambda_2 = 0$, and that the approximately equals sign is actually an equals sign. Find the rate of change of $\bar{t}$ with respect to γ.

Continued on next page

4. The resistance R of the flow of blood through a blood vessel (assumed to have the shape of a cylindrical tube) is given by

$$R = \frac{Kl(\gamma + 1)^2}{d^4}$$

where l is the length of the tube, d is its diameter and $\gamma \geq 0$ is the curvature. The positive constant K represents the viscosity of the blood (viscosity is a measure of the resistance of fluid to stress; water has low viscosity, honey has high viscosity).

(a) Find the derivative of R with respect to l and interpret your answer. Does it make sense?

(b) Find the derivative of R with respect to d and interpret your answer. Does it make sense?

THE END

Math functions in context – Integration

1. Find the following integrals. Keep in mind that the notation for the integral tells you what the variable is.

(a) $\int ae^{bx}\,dx$

(b) $\int ae^{bt}\,dt$

(c) $\int Me^{-(b+c)t}\,dt$

(d) $\int \left(\gamma b^2t^2 - abt\right)\,dt$

(e) $\int (N_eV_e + N_iV_i)\,dN_i$

(f) $\int \frac{N_eV_e + N_iV_i}{4}\,dV_i$

(g) $\int \frac{7}{aMt}\,dt$

Continued on next page

(h) $\int \frac{Kl(\gamma+1)^2}{D^4}\, dK$

(i) $\int \frac{Kl(\gamma+1)^2}{D^4}\, d\gamma$

(j) $\int \frac{Kl(\gamma+1)^2}{D^4}\, dD$

(k) $\int b\left(e^{-at} + Ne^{t}\right)\, dt$

(l) $\int b\left(e^{-xt} + xe^{t}\right)\, dt$

(m) $\int b\left(e^{-xt} + xe^{t}\right)\, dx$

(n) $\int b\left(e^{-xt} + xe^{t}\right)\, db$

Continued on next page

2. According to Von Bertalanffy model, the rate of growth of pacific salmon is given by $dL/dt = 12.6e^{-0.15t}$, where L is in centimetres and t is in years.

(a) Given that $L(0) = 0$ (i.e., at the moment of fertilization it is assumed that the length is zero), find a formula for the length $L(t)$ of pacific salmon.

(b) Use (a) to find how much the salmon grows between the ages of 2 and 3.

(c) Use the definite integral to compute how much the salmon grows between the ages of 2 and 5.

Continued on next page

3. Most human papillomavirus (HPV) infections in young women are temporary and have very little long-term effects. Assume that $P(t)$ is the proportion (or percent) of women initially infected with the HPV who no longer have the virus at time t (t is measured in years). The *rate of change* of $P(t)$ is modelled by the function

$$p(t) = 0.5 - 0.21t^{1.5} + 0.5e^{-1.2t},$$

where $0 \leq t \leq 2$.

(a) For what proportion/percent of young women will the virus be gone within one year?

(b) For what proportion/percent of young women will the virus be gone within two years?

Continued on next page

4. Assume that $P(t)$ is the proportion of women initially infected with the HPV who no longer have the virus at time t (t is measured in years). Another model for the *rate of change* of $P(t)$ is given by

$$p(t) = 1.7e^{-1.5t} - \frac{0.0016}{(t+0.01)^2}$$

where $0 \leq t \leq 10$.

(a) For what proportion of young women will the virus be gone within one year?

(b) What proportion of women will still have the virus after 10 years?

Continued on next page

5. The rate of change of the number of new cases infected by a strain H2T1 of influenza A virus is given by $dv/dt = 245.1(t-1)^2$, where t is time in months; the time $t = 1$ represents January 2013. It is known that in January 2013 there were 200 cases of flu.

(a) Find a formula for $v(t)$.

(b) According to this model, how many people will be infected by the end of the year (on top of the initial 200 cases)?

(c) Explain why this model cannot be used for long term predictions (i.e., 50 or 100 years from now).

THE END

MATHEMATICS 1LS3 TEST 1

Day Class

E. Clements, J. Hofscheier, M. Lovrić

Duration of Examination: 60 minutes

McMaster University, 1 October 2018

First name (PLEASE PRINT): ____________________

Family name (PLEASE PRINT): ____________________

Student No.: ____________________

THIS TEST HAS 8 PAGES AND 6 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

Total number of points is 40. Marks are indicated next to the problem number. Calculator allowed: McMaster standard calculator Casio fx991MS or Casio fx991MS PLUS or lower Casio which has two lines of display and no graphing capabilities.

EXCEPT ON QUESTIONS 1 AND 2, you must show work to receive full credit.

Problem	Points	Mark
1	10	
2	6	
3	4	
4	7	
5	7	
6	6	
TOTAL	40	

Continued on next page

Name: ____________________

Student No.: ____________________

1. **Multiple choice questions: circle ONE answer. No justification is needed.**

(a)[2] The average half-life of acetaminophen (active ingredient in tylenol) is 2.5 hours. Assume that a patient is given a dose of 1000 mg of acetaminophen.

Identify all correct statements.

(I) After 5 hours, 250 mg of acetaminophen is left unabsorbed in patient's body.
(II) After 2 hours, 450 mg of acetaminophen is left unabsorbed in patient's body.
(III) After 10 hours, less than 100 mg of acetaminophen is left unabsorbed in patient's body.

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

(b)[2] The resistance R of the flow of blood through a curved blood vessel (assumed to have the shape of a cylindrical tube) is given by

$$R = \frac{Kl\gamma^2}{d^4}$$

where l is the length of the tube, d is its diameter and $\gamma \geq 1$ is the curvature. The positive constant K represents the viscosity of the blood (viscosity is a measure of the resistance of fluid to stress; water has low viscosity, honey has high viscosity).

Identify all correct statements.

(I) If the curvature γ doubles, then the resistance R doubles.

(II) If the viscosity K doubles, then the resistance R doubles.

(III) If the diameter d doubles, then the resistance R increases $2^4 = 16$-fold.

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

Continued on next page

(c)[2] What is the domain of the function $f(x) = \ln(5 - x) + \log_{10}(x + 2)$?

(A) $x < 0$ (B) $x < -2$ (C) $x < 5$ (D) $x < -2$ and $x > 5$
(E) $-2 < x < 0$ (F) $-2 < x < 5$ (G) $0 < x < 5$ (H) $x < -2$ and $x > 0$

(d)[2] Identify all correct statements about the function $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$.

(I) $f(x)$ is continuous at $x = 0$.
(II) $x = -1$ is a vertical asymptote of the graph of $f(x)$.
(III) $y = 1$ is a horizontal asymptote of the graph of $f(x)$.

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

(e)[2] The formula (adapted from M. Benton and D. Harper, *Basic Palaeontology*)

$$\text{Sk} = e^{0.08T}\,\text{Sp}^{0.84}$$

relates the skull length, Sk, of a larger dinosaur to its spine length, Sp, at the time of death T.
Identify all correct statements.

(I) The semilog graph of Sk as a function of Sp is a line.

(II) The semilog graph of Sk as a function of T is a line.

(III) The double log graph of Sk as a function of Sp is a line.

(Note: It does not matter whether ln or log is used.)

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

Continued on next page

Name: ____________________
Student No.: ____________________

2. **True/false questions: circle ONE answer. No justification is needed.**

(a)[2] The formula $H = Mx+4$, where M is a constant, represents a proportional relationship between H and x.

TRUE FALSE

(b)[2] An oscillatory input (intensity function of a group of spiking neurons) is given by the formula $\hat{\lambda}_1(t) = v_0 + a\cos(2\pi f_m(t+d))$. The period of $\hat{\lambda}_1(t)$ is f_m.

TRUE FALSE

(c)[2] The limit $\displaystyle\lim_{x\to 0} \frac{|x-1|}{x-1}$ is a real number.

TRUE FALSE

Continued on next page

Questions 3-6: You must show correct work to receive full credit.

3. The visibility index tells us how clearly we can see an object submerged in water. In saltwater, the visibility index is given by the function

$$v(d) = \frac{1.75}{0.79 + \ln(8.3d + 2.61)}$$

where d is the depth in metres, $0 \leq d \leq 50$ (so $d = 0$ labels the surface, and $d = 3$ is 3 m below the surface).

(a)[1] State (in one sentence) what question related to the visibility index is answered by finding the inverse function.

(b)[3] Find the inverse function of $v(d)$.

Continued on next page

4. (a)[2] What is the domain of the function $f(x) = 9.71 - \arcsin(2x - 5.4)$?

(b)[3] The body mass index is defined as BMI=m/h^2 where the mass m is in kilograms and the height h in metres. Find a formula for the body mass index in the case where the mass is in kilograms, but the height is measured in centimetres.

(c)[2] Consider the data in the table. Which coordinate system (the usual xy-coordinate system, semilog or double-log) is the most suitable to represent this data set? Why?

x	y
0.05	4
0.1	18
1	4400
20	2.6
240	0.1
2500	0.0029

Continued on next page

5. (a)[3] Consider the formula for human population growth

$$P(t) = 4.43\left(\frac{\pi}{2} + \arctan\frac{t-2007}{42}\right)$$

where t is a calendar year and $P(t)$ is in billions. Find the range of $P(t)$. Based on it, state the maximum world population (two decimal places suffice) predicted by this model.

(b)[4] A population of river sharks (freshwater sharks) in New Zealand changes periodically with a period of 12 months, and is measured at the start of each month. In January, it reaches a maximum of 12,600, and in July it reaches a minimum of 5,800. By selecting an appropriate trigonometric function, find a formula which describes how the population of river sharks changes with time.

Continued on next page

Name:____________________

Student No.: ____________________

6. (a)[3] Find $\lim_{x\to\infty}\left(\ln(2x^3+4)-\ln(x^2+x^3+2)\right)$ or else say that it does not exist.

(b)[3] Identify all x for which the function $f(x)=\dfrac{\sqrt{e^{3-x}-10}}{x}$ is continuous.

THE END

MATHEMATICS 1LS3 TEST 2

Day Class
Duration of Examination: 60 minutes
McMaster University, 29 October 2018

E. Clements, J. Hofscheier, M. Lovrić

First name (PLEASE PRINT): ____________

Family name (PLEASE PRINT): ____________

Student No.: ____________

THIS TEST HAS 8 PAGES AND 6 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. USE A PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

Total number of points is 40. Marks are indicated next to the problem number. Calculator allowed: McMaster standard calculator Casio fx991MS or Casio fx991MS PLUS or lower Casio which has two lines of display and no graphing capabilities.

EXCEPT ON QUESTIONS 1 AND 2, you must show work to receive full credit.

Problem	Points	Mark
1	10	
2	6	
3	6	
4	6	
5	6	
6	6	
TOTAL	40	

Continued on next page

Name:______________________

Student No.: ____________________

1. **Multiple choice questions: circle ONE answer. No justification is needed.**

(a)[2] It is known that $f(4) = 0$, $f'(4) = 0$ and $f''(4) = 0$. Which statement(s) is/are true for all functions $f(x)$ which satisfy these two conditions?

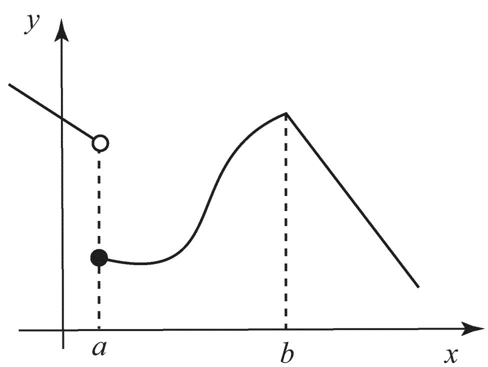
(I) $f(4) = 0$ is a local (relative) minimum of $f(x)$

(II) the tangent line to the graph of $f(x)$ at $x = 4$ is $y = 0$

(III) $f(4) = 0$ is a point of inflection of the graph of $f(x)$

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

(b)[2] Identify all correct statements for the function $f(x)$ whose graph is given below.



(I) $f(x)$ is differentiable at a

(II) $f(x)$ is continuous at b

(III) $f(x)$ is differentiable at b

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

Continued on next page

(c)[2] The slope of the tangent to the curve given implicitly by $x^2y^4 = 1$ at the point $(1, 1)$ is

(A) 2	(B) -2	(C) 1	(D) -1
(E) $-1/4$	(F) $-1/2$	(G) $1/2$	(H) $1/4$

(d)[2] If $f(x) = Ax\ln(B + x)$, then $f'(0)$ is equal to

(A) A	(B) B	(C) AB	(D) $B\ln B$
(E) $AB\ln B$	(F) $AB\ln A$	(G) $A\ln B$	(H) $B\ln A$

(e)[2] Identify all correct Taylor polynomials of the function $f(x) = \sin 2x$ at $x = 0$.

(I) $T_1(x) = x$

(II) $T_3(x) = 2x - \dfrac{x^3}{3}$

(III) $T_3(x) = 2x - \dfrac{4x^3}{3}$

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

Continued on next page

Name:________________
Student No.: ________________

2. **True/false questions: circle ONE answer. No justification is needed.**

(a)[2] Knowing that $g''(x) = x \ln x$, we conclude that the function $g(x)$ is concave down on $(0, 1)$.

TRUE FALSE

(b)[2] The linear-quadratic model for the percent S of cancer cells surviving radiation treatment states that

$$S(d) = e^{-d^2 - 0.1d - 0.2}$$

where $d \geq 0$ is the dose (in Gray) per treatment of radiation.
$S(d)$ is an increasing function.

TRUE FALSE

(c)[2] Let $m(t)$ represent the mass of melting snow in kilograms, where t is the time in days. The units of $m'(t)$ are kilograms.

TRUE FALSE

Continued on next page

Name:____________________

Student No.: ____________________

Questions 3-6: You must show correct work to receive full credit.

3. (a)[3] Find $\lim_{x\to 0} \frac{\sin x - x}{x^3}$

(b)[3] Find $\lim_{x\to 0^+} x^4 \ln x$

Continued on next page

4. (a)[3] The resistance R of the flow of blood through a blood vessel (assumed to have the shape of a cylindrical tube) is given by

$$R = \frac{K^{0.96} L(\gamma + 1)^2}{d^4}$$

where L is the length of the tube, d is its diameter and $\gamma \geq 0$ is the curvature. The positive constant K represents the viscosity of the blood.

Find the derivative of R with respect to d and interpret your answer, i.e., explain what your answer implies for the dependence of R on the diameter of a blood vessel.

(b)[3] In the article *Phenomenological Theory of World Population Growth* by S. Kapitza, Physics-Uspekhi (39)1, we find the formula

$$P(t) = 4.43\left(\frac{\pi}{2} + \arctan\frac{t}{42}\right)$$

where t is the time in years, with $t = 0$ representing 2007.

Find the linear approximation of $P(t)$ at $t = 0$. Round off all numbers to two decimal places.

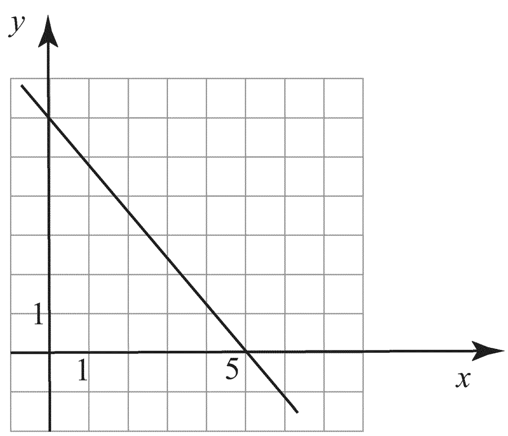
Continued on next page

5. (a)[3] In *Hybrid equation/agent-based model of ischemia-induced hyperemia and pressure ulcer formation* by Alexey Solovyev et al., PLoS Computational Biology 9.5 (May 2013), the authors analyze the function

$$I(t) = I_{rest}\left(1 + ae^{-2t} + be^{-3t}\right)$$

where I_{rest} and a are positive constants, and the parameter b is negative. Find all critical numbers (t values only) of $I(t)$.

(b)[3] Let $h(x) = x\sin(f(x))$. The graph of $f(x)$ is a line shown below. Find $h'(5)$.



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6. (a)[2] Show that $f(x) = (x^2 - 1)e^{-x^2}$ has three critical points 0, $-\sqrt{2}$, and $\sqrt{2}$.

(b)[2] State the Extreme Value Theorem. Make sure to clearly identify assumption(s) and conclusion(s).

(c)[2] Find the absolute maximum and the absolute minimum of the function $f(x)$ from (a) on the interval $[0, 2]$.

THE END

MATHEMATICS 1LS3 TEST 3

Day Class E. Clements, J. Hofscheier, M. Lovrić
Duration of Examination: 60 minutes
McMaster University, 26 November 2018

First name (PLEASE PRINT): ____________________

Family name (PLEASE PRINT): ____________________

Student No.: ____________________

THIS TEST HAS 8 PAGES AND 6 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. USE A PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

Total number of points is 40. Marks are indicated next to the problem number. Calculator allowed: McMaster standard calculator Casio fx991MS or Casio fx991MS PLUS or lower Casio which has two lines of display and no graphing capabilities.

EXCEPT ON QUESTIONS 1 AND 2, you must show work to receive full credit.

Problem	Points	Mark
1	10	
2	6	
3	5	
4	5	
5	7	
6	7	
TOTAL	40	

Continued on next page

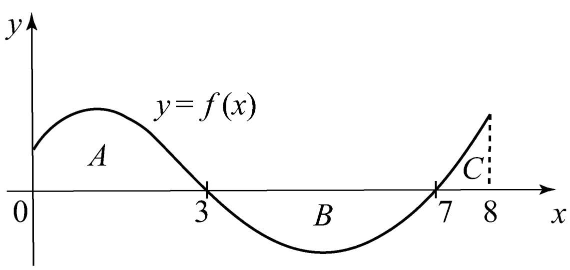
1. **Multiple choice questions: circle ONE answer. No justification is needed.**

(a)[2] $\int_1^2 \frac{1}{(x-3)^2}\,dx =$

(A) 0 (B) $-1/2$ (C) $-1/3$ (D) $1/2$

(E) $1/3$ (F) $3/2$ (G) $-3/2$ (H) $2/3$

(b)[2] In the graph below, the area of A is 4, the area of B is 6 and the area of C is 1 (in some units squared). Identify all correct statements.



(I) $\int_0^8 f(x)\,dx = -1$ (II) $\int_3^8 2f(x)\,dx = -5$ (III) $\int_3^7 (f(x)+3)\,dx = 6$

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

Continued on next page

(c)[2] Which of the following definite integral(s) is/are positive? (Hint: Think! No need to calculate the integrals.)

$$\text{(I)} \int_0^2 \cos x\, dx \qquad \text{(II)} \int_0^3 \cos x\, dx \qquad \text{(III)} \int_0^4 \cos x\, dx$$

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

(d)[2] The average value of $f(x) = \sin x$ on $[0, \pi]$ is

(A) 0 (B) 1 (C) π (D) $\pi/2$
(E) $\pi/4$ (F) $2/\pi$ (G) $1/\pi$ (H) $\pi/8$

(e)[2] Which of the following improper integrals are *convergent*?

$$\text{(I)} \int_1^\infty x^{-1.8}\, dx \qquad \text{(II)} \int_1^\infty x^{-1}\, dx \qquad \text{(III)} \int_1^\infty x^{-0.11}\, dx$$

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

Continued on next page

Name:__________________

Student No.: __________________

2. **True/false questions: circle ONE answer. No justification is needed.**

(a)[2] $P(t) = 9e^{0.1t}$ is a solution of the initial value problem $P'(t) = 0.9P(t)$, $P(0) = 10$.

TRUE FALSE

(b)[2] It is known that $\int_1^6 f(x)\,dx = -10$. Thus, $f(x) < 0$ for all x in $[1, 6]$.

TRUE FALSE

(c)[2] The left and the midpoint Riemann sums of $f(x) = x^{-1/3}$ on $[2, 12]$ satisfy $M_{15} < L_{15}$.

TRUE FALSE

Continued on next page

Questions 3-6: You must show correct work to receive full credit.

3. In December 2016, there was a notable increase in influenza cases (caused by the H3N2-like virus) in Winnipeg. Some researchers suggested that the number of influenza cases in Winnipeg could be modelled by

$$I'(t) = 240e^{-0.5t} - 40e^{-0.8t}$$

where t is time in days, with $t = 0$ representing 12 December 2016. On 12 December 2016, there were 230 reported cases of influenza in Winnipeg.

(a) [2] Estimate the number of influenza cases in Winnipeg on 14 December 2016 using Euler's Method with a step size of $\triangle t = 2$. Round your answer to the nearest integer.

(b) [3] Find a formula for $I(t)$ algebraically and use this formula to find the actual number of influenza cases in Winnipeg on 14 December 2016. Round your answer to the nearest integer.

Continued on next page

 Name:________________
Student No.: ________________

4. (a) [2] Find an approximation of the area of the region below the graph of $y = \ln x$ and over the interval $[1, 3]$, using a Riemann sum with 4 rectangles and right endpoints. Round your answer to three decimal places. Sketch the function and the four rectangles involved.

(b) [3] Find the exact area of the region in part (a) by evaluating $\int_1^3 \ln x \, dx$. Round your answer to three decimal places.

Continued on next page

5. (a)[3] Sketch (shade) the region bounded by the graphs of $y = \sqrt{x}$, $y = 1$, $x = 0$ and $x = 4$. Set up, but **do not evaluate**, the formula for its area. Your formula should not include absolute value.

(b)[4] Set up a formula for the volume of the solid obtained by rotating the region in part (a) about the y-axis. Your formula should not include absolute value. **Do not evaluate the integral.**

Continued on next page

Name:________________
Student No.: ________________

6. A blood concentration of acetaminophen (common pain reliever) higher than 200 mcg/mL (micrograms per millilitre), reached 4 hours after ingestion, is known to increase the risk of liver damage. Even without taking a medication, a small amount of acetaminophen can be found in the body; consequently, we assume that the initial concentration is 8 mcg/mL. (Source: University of Rochester Medical Centre.)

Suppose that the concentration of acetaminophen in the blood of a patient, when following a certain protocol (such as after a minor surgery), changes according to $c'(t) = 40.2te^{-0.1t^2}$, measured in mcg/mL per hour. The dosing protocol (ingestion) starts when $t = 0$.

(a) [3] Find the indefinite integral $\int 40.2te^{-0.1t^2}\,dt$.

(b) [2] Determine whether a patient, subjected to the dosing protocol described above, faces an increased risk of liver damage.

(c) [2] What does the integral $\int_0^\infty 40.2te^{-0.1t^2}\,dt$ represent, and what are its units?

THE END

MATHEMATICS 1LS3 FINAL EXAMINATION
version 1

Day Class
Duration of Examination: 2.5 hours
McMaster University

FIRST NAME (PRINT CLEARLY): ____________

FAMILY NAME (PRINT CLEARLY): ____________

Student No.: ____________

THIS EXAMINATION PAPER HAS 15 PAGES AND 8 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF THE INVIGILATOR.

Total number of points is 75. Marks are indicated next to the problem number. McMaster Standard Calculator Casio fx991MS or Casio fx991MS+ is allowed. Write your answers in the space provided. EXCEPT ON QUESTIONS 1 AND 2, YOU MUST SHOW WORK TO GET FULL CREDIT. Good luck.

Problem	Points	Mark
1	20	
2	6	
3	12	
4	7	
5	8	
6	6	
7	7	
8	9	
TOTAL	75	

Question 1: Circle the correct answer. No justification is needed.

1. (a)[2] The resistance R of the flow of blood through a blood vessel (assumed to have the shape of a cylindrical tube) is given by

$$R = \frac{K l \gamma^2}{d^4}$$

where l is the length of the tube, d is its diameter and $\gamma \geq 1$ is the curvature. The positive constant K represents the viscosity of the blood (viscosity is a measure of the resistance of fluid to stress; water has low viscosity, honey has high viscosity).

Which of the following statements is/are true?

(I) If the curvature γ doubles, then the resistance R doubles

(II) If the viscosity K doubles, then the resistance R doubles

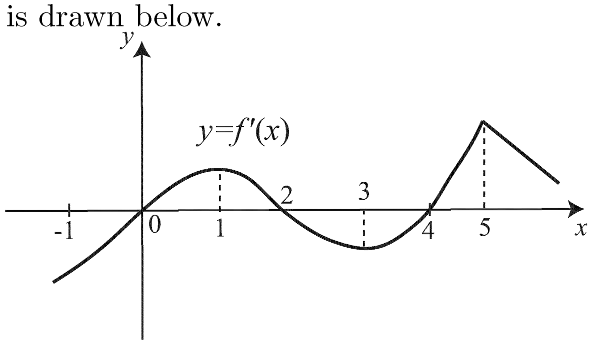
(III) If the diameter d doubles, then the resistance R increases $2^4 = 16$ fold

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

(b)[2] If $g(x) = \cos\left(e^{x^2+x}\right)$, then $g'(0)$ is equal to

(A) 0	(B) $\sin 1$	(C) $\cos 1$	(D) $\sin e$
(E) $-\sin 1$	(F) $\cos e$	(G) $-\cos e$	(H) $-\cos 1$

(c)[2] Determine which of the statements is/are true for the function $f(x)$ whose **derivative** $f'(x)$ is drawn below.



(I) $x = 3$ is a critical point (critical number) of $f(x)$

(II) $x = 4$ is a critical point (critical number) of $f(x)$

(III) $x = 5$ is a critical point (critical number) of $f(x)$

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

(d)[2] $\lim_{x \to 0^+} \sqrt{x}\, \ln(3x) =$

(A) 3	(B) ∞	(C) $-\infty$	(D) -3
(E) 0	(F) 1/3	(G) $-1/3$	(H) 1/2

(e)[2] The formula (adapted from M. Benton and D. Harper, *Basic Palaeontology*)

$$\text{Sk} = e^{0.08T}\,\text{Sp}^{0.84}$$

relates the skull length, Sk, of a larger dinosaur to its spine length, Sp, at time of death T. Which of the following statements is/are true?

(I) The semilog graph of Sk as a function of Sp is a line

(II) The semilog graph of Sk as a function of T is a line

(III) The double log graph of Sk as a function of Sp is a line

(Note: it does not matter whether ln or log is used.)

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

(f)[2] Identify all correct Taylor polynomials of the function $f(x) = \sin x$ at $x = 0$.

(I) $T_1(x) = x$

(II) $T_3(x) = x - x^2$

(III) $T_3(x) = x - \dfrac{x^3}{2}$

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

(g)[2] Identify all correct interpretations of the definite integral $\int_2^3 (2x-1)\,dx$.

(I) Total change in the function $f(x) = 2x - 1$ from $x = 2$ to $x = 3$.

(II) Area of the region under the graph of $f(x) = 2x - 1$, above the x-axis, from $x = 2$ to $x = 3$.

(III) Average value of the function $f(x) = 2x - 1$ on the interval $[2, 3]$.

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

(h)[2] Consider the differential equation

$$P'(t) = 1.1P(t)\left(1 - \frac{P(t)}{1400}\right)\left(1 - \frac{650}{P(t)}\right)$$

where $P(t)$ represents the number of elk in Douglas Provincial Park in Saskatchewan. The variable t represents time in years, with $t = 0$ representing 2013.
What does this model say about the population of elk? (Identify all true statements.)

(I) At the moment when there are 1250 elk, the population is increasing

(II) At the moment when there are 1450 elk, the population is decreasing

(III) At the moment when there are 600 elk, the population is increasing

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

(i)[2] Start with the graph of $y = \cos x$. Scale (expand) the graph horizontally by a factor of 3 and then shift right the graph you obtained by 6 units. Finally, expand this graph vertically by a factor of 4. The graph you obtained is

(A) $y = \frac{1}{4}\cos\left(\frac{x+2}{6}\right)$ (B) $y = \frac{1}{4}\cos\left(\frac{x-2}{6}\right)$ (C) $y = 4\cos\left(\frac{x+6}{3}\right)$

(D) $y = 4\cos\left(\frac{x}{3} - 2\right)$ (E) $y = 4\cos\left(\frac{x}{3} + 6\right)$ (F) $y = 4\cos\left(\frac{x}{3} - \frac{2}{3}\right)$

(G) $y = \frac{1}{4}\cos\left(\frac{x+2}{3}\right)$ (H) $y = \frac{1}{4}\cos\left(\frac{x-6}{3}\right)$

(j)[2] Which of the following improper integrals is/are convergent?

(I) $\int_0^1 x^{-2}\,dx$ (II) $\int_0^1 x^{-1}\,dx$ (III) $\int_0^1 x^{-0.5}\,dx$

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

Question 2: Circle the correct answer. No justification is needed.

2. (a)[2] Based on knowing that $\lim_{x \to 2} f(x) = 4$, we can conclude that the function $f(x)$ is continuous at 2, and $f(2) = 4$.

TRUE FALSE

(b)[2] A population of monkeys on an island doubles every year; present count is 25 monkeys. Based on this data, we conclude that in 3 years there will be 75 monkeys on the island.

TRUE FALSE

(c)[2] A population of bacteria triples every hour. Every hour, after reproduction, 800 bacteria are removed. The population starts with 1000 bacteria. The dynamical system which describes this population is given by $b_{t+1} = 3(b_t - 800)$, $b_0 = 1000$.

TRUE FALSE

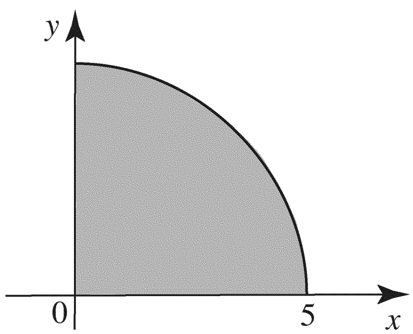
Questions 3-8: You must show work to obtain full credit

3. (a)[3] Find $\int x \ln x dx$.

(b)[3] Find $\int_1^9 \frac{(1+\sqrt{x})^2}{\sqrt{x}}\, dx$.

(c)[3] Consider the region R bounded by $y = \sqrt{x+4}$, $y = 3$, $x = 0$, and $x = 6$. Write a formula for the volume of the solid obtained by rotating the region R about the x-axis. Your answer must not contain absolute value. You do NOT need to compute the volume.

(d)[3] Using a definite integral, write an expression for the area of the part of the circle of radius 5 centered at the origin lying in the first quadrant, as shown below. You do NOT need to compute the integral.



4. Consider the function $f(x) = x^{4/5}(2x - 9)$.

(a)[3] Show that $x = 0$ and $x = 2$ are the only critical points of $f(x)$.

(b)[2] Give a precise statement of the Extreme Value Theorem.

(c)[2] Find the absolute maximum and the absolute minimum values of $f(x)$ on $[1, 5]$.

5. The rate at which new influenza cases occured in 2013 in Greater Edmonton Area followed the formula $dI/dt = 24e^{-0.2t} - 21e^{-0.3t}$. Units of $I(t)$ are *thousands* of people. By t we represent the time in days measured from 1 December 2013 (so $t = 0$ represents 1 December 2013). On 1 December 2013 there were 1250 cases of influenza.

(a) [4] On which day (state the date) did the rate dI/dt reach its largest value?

(b)[1] Write the initial value problem for the number $I(t)$ of influenza cases. Keep in mind that the number of people is measured in thousands.

(c)[3] Solve the initial value problem in (b) to find the formula for $I(t)$.

6. Consider the alcohol consumption model $a_{t+1} = a_t - \dfrac{9.1a_t}{3.2 + 1.1a_t} + d$, where a_t is the amount of alcohol (in grams) and d is the constant amount that is consumed every hour.

(a)[1] What is the meaning of the term $\dfrac{9.1a_t}{3.2 + 1.1a_t}$?

(b)[2] Assume that $d = 5$. Find the equilibrium point of the given system.

(c)[3] Assume that $d = 5$. Determine whether the equilibrium you found in (b) is stable or unstable.

7. Most human papillomavirus (HPV) infections in young women are temporary and have very little long-term effects. Assume that $P(t)$ is the proportion (or percent) of women initially infected with the HPV who no longer have the virus at time t (t is measured in years). The *rate of change* of $P(t)$ is modelled by the function

$$p(t) = 0.5 - 0.25t^{1.5} + 0.5e^{-1.2t}$$

where $0 \leq t \leq 2$.

(a)[2] Compute $p(1)$ to two decimal places. What are the units of $p(1)$?

(b)[3] Find $\int_0^1 \left(0.5 - 0.25t^{1.5} + 0.5e^{-1.2t}\right) dt$. Round off to two decimal places.

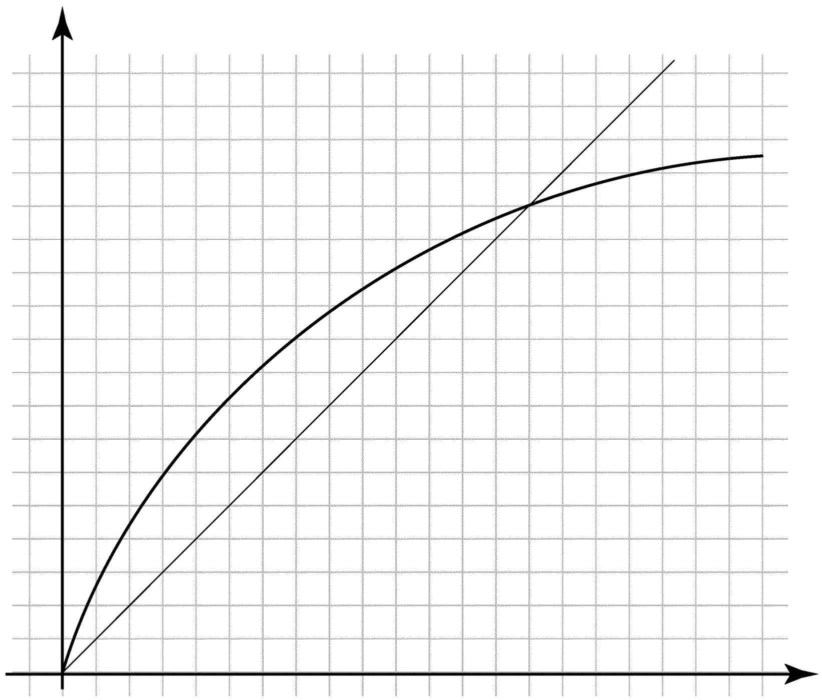
(c)[2] What does the number you obtained in (b) represent in the context of HPV infections?

8. Consider the discrete-time dynamical system $p_{t+1} = \dfrac{3.4p_t}{2.7 + 0.001p_t}$, where p_t represents a number of frogs in thousands and time is in years.

(a)[3] Identify the per capita production function. Sketch its graph and label intercepts. Explain why the per capita production function makes sense.

(b)[2] Find all equilibrium points.

(c)[2] The graph below shows the updating function of the given dynamical system. Label the coordinate axes and label all equilibrium points. Using cobwebbing, determine whether the smallest equilibrium point is stable or unstable.



(d)[2] Explain what the stability or instability of the smallest equilibrium point means for the population of frogs.

Math 1LS3

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